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Publication date:
1985

subfaculteit der econometrie

RESEARCH MEMORANDUM

TILBURG UNIVERSITY
DEPARTMENT OF ECONOMICS
Postbus 90153 - 5000 LE Tilburg
Netherlands

FEW
196

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november 1985

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This research is part of the VF-program "Equilibrium and Disequilibrium in Demand and Supply", which has been approved by the Dutch Office of Education and Sciences.

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Abstract

In this paper three different adjustment processes for finding solutions to the nonlinear complementarity problem on the product space S of several unit simplices (NLCP on S) are discussed. Since for example the problems of finding equilibria in noncooperative N-person games and international trade models can be transformed to an NLCP on S, these processes are suitable for finding Nash equilibria and equilibrium prices. It appears that these processes posses a transparant dynamic interpretation, while they converge under very weak conditions. The latter is due to the fact that the processes always keep the starting point in mind, which prevents them from cycling or leaving the simplotope. In that sense the processes might be preferred above the Walrasian and global Newton processes which converge only under rather strong conditions. Another advantage of the processes is that they can be followed discretely and arbitrary close by so-called simplicial algorithms.

Keywords: equilibrium, adjustment process, complementarity

1. Introduction

In a Walrasian economy with n+1 commodities the solution space is the n-dimensional unit simplex of price vectors

$$S^n = \{p \in R_+^{n+1} | \Sigma_{j=1}^{n+1} p_j = 1\}.$$

In an earlier paper [15], we have dealt with convergent adjustment processes for finding an equilibrium price vector on the unit simplex. Here we are concerned with convergent processes on the simplotope S being the product space of several unit simplices. For example, in the noncooperative N-person game the solution space is the simplotope

$$S = \Pi_{j=1}^{N} S^{n_j}$$

where n_j+1 is the number of strategies of player j, j = 1,...,N. A strategy vector is an element $x = (x_1^T,\ldots,x_N^T)^T$ in S with $x_j \in S^{n_j}$ the strategy vector of player j. Another example concerns the international trade model with domestic goods traded within one country only and internationally traded common goods. As has been shown by van der Laan [11], the price space can be formulated as the simplotope S with S^{n_N} the price simplex of the n_N+1 common goods and with S^{n_j}, j = 1,...,N-1, the set of elements x^j determining in country j both the (relative) prices of the n_j domestic goods and the price level of these goods with respect to the price level of the common goods.

Our purpose is to give convergent adjustment processes which lead to a solution to problems formulated on the simplotope. All the processes to be discussed in this paper are generalizations of processes on the unit simplex S^n which are based on simplicial pivoting algorithms.

Starting with the work of Scarf [17, 18] and Kuhn [9, 10] several (simplicial) pivoting algorithms have been developed to solve the problem of finding a vector of equilibrium prices in S^n for a Walrasian economy with n+1 commodities. Amongst these are the variable dimension restart algorithms of van der Laan and Talman [12], Doup and Talman [1] and Doup, van der Laan and Talman [2]. These algorithms approximately follow paths of prices by generating a sequence of simplices of varying dimension in a simplicial subdivision of the n-dimensional unit simplex S^n. In [15] it is shown that these paths of prices can be seen as convergent processes leading from an arbitrarily chosen price vector to an equilibrium price vector. In this sense these paths can serve as an alternative for the classical Walras tatonnement process or the global Newton price adjustment process (see Smale [19]).

The simplicial variable dimension restart algorithms on S^n differ from each other in the number of rays along which the arbitrarily chosen starting point v can be left and the directions in which these rays are pointing. In [12] and [13] there are n+1 rays. To each of the n+1 facets $\{p \in S^n | p_i = 0\}$, $i = 1,\ldots,n+1$, of S^n just one ray is pointing. In the algorithm on S^n given in [1] there are also n+1 rays, but now pointing from v to the n+1 vertices of S^n. Finally, in [2] an algorithm has been developed in which there is a ray to each of the $2^{n+1}-2$ faces $\{p \in S^n | p_i = 0 \text{ if } i \in T\}$ of S^n with T any nonempty proper subset of the set of indices $I_{n+1} = \{1,\ldots,n+1\}$. Leaving the starting point v along a ray corresponding to some subset T, the prices of the commodities j in T are all decreased while the prices of the commodities j not in T are all increased. Taking T such that $j \in T$ if the excess demand of good j at v is negative and j not in T if it is positive, it follows that from the starting point the prices of the commodities with positive excess demand are increased while the prices of the commodities with negative excess demand are decreased. This is closely related to the classical tatonnement process. For the existence proofs of the three processes on S^n and for further details we refer to [15].

In this paper we prove the existence of paths leading from an arbitrarily chosen starting point v in the simplotope S to a solution point x^* in S for which $z(x^*) \leq 0$, where z is a continuously differentiable function on the simplotope $S = \Pi_{j=1}^{N} S^{n_j}$. With $z(x) = (z_1^T(x),\ldots,z_N^T(x))^T$ we assume that for all $x \in S$ and $j \in I_N$ both $z_j(x) \in R^{n_j+1}$ and $x_j^T z_j(x) = 0$.

First we consider the existence of paths induced by the generalizations of the two (n+1)-ray algorithms on S^n. The first generalization is due to van der Laan and Talman [14], see also Talman [20] and van der Laan, Talman and Van der Heyden [16]. They developed a simplicial variable dimension algorithm on S with $\Sigma_{j=1}^{N}(n_j+1)$ rays to leave the starting point. Observe that the number of rays equals the number of facets of S. A variable dimension algorithm on S with $\Pi_{j=1}^{N}(n_j+1)$ rays was developed in Doup and Talman [1]. In this case the number of rays equals the number of vertices of S. Although for the unit simplex the number of vertices equals the number of facets, the difference between the two algorithms on S is similar to the difference of the two corresponding (n+1)-ray algorithms on S^n given in [12] and [1]. Their generalizations to S will be called the sum- and the product-ray algorithm respectively. The corresponding processes which approximately describe the paths followed by the algorithms are called the sum- and the product-process.

Besides describing the processes for finding equilibria on S which are induced by known algorithms on S, we will also present a path following process in which the starting point v can be left along $\Pi_{j=1}^{N}(2^{n_j+1}-2)$ rays. This process is a generalization of the process on S^n having $2^{n+1}-2$ rays. For a simplicial variable dimension algorithm to follow approximately the latter path on S^n we refer to [2]. We will call this process on S the exponent-process. When applied to find equilibrium strategy vectors in a noncooperative game the path seems to have a plausible strategic interpretation in the sense that initially the probabilities of the profitable strategies are increased whereas the probabilities of the unprofitable strategies are decreased. At first glance the path is very similar to the classic Walras tatonnement process when applied to economic models. However, convergence is assured by the fact that the process keeps the starting point v in mind.

The paper is organized as follows. In section 2 we describe both the sum- and the product-process on S. The exponent-process on S is presented in section 3. An explanation of these processes in terms of adjustment processes is given in section 4. The existence proofs of the paths are postponed till the sections 5 and 6. For these proofs we need the concept of primal-dual manifolds as introduced by Kojima and Yamamoto [7]. This theory is summarized at the beginning of section 5. The remainder of section 5 is devoted to the existence proof of the paths generated by the sum- and the product-process. Finally, in section 6 the convergence and existence of the path belonging to the exponent-process on S is proved.

2. The sum- and the product-process on S

In this section we describe two convergent adjustment processes for finding a solution point $x^* \in S$ with $z(x^*) \leq 0$, where z is a continuously differentiable function from the simplotope $S = \Pi_{j=1}^{N} S^{n_j}$ to the product space $\Pi_{j=1}^{N} R^{n_j+1}$, such that for all $x = (x_1^T,\ldots,x_N^T)^T \in S$, $z(x) = (z_1^T(x),\ldots,z_N^T(x))^T$ satisfies

$$x_j^T z_j(x) = 0 \qquad \text{for all } j = 1,\ldots,N.$$

These processes will be called the sum- and the product-process and correspond to the simplicial variable dimension algorithms on S of van der Laan and Talman [14] and Doup and Talman [1] respectively. The processes correspond to the algorithms in the sense that the path of points traced by the processes can be followed approximately by these algorithms. In fact, the paths can be followed arbitrarily close by taking the mesh of the underlying triangulation small enough. We can therefore consider the path traced by the process as the limiting path of the corresponding algorithm, being the path of points traced by the algorithm when the mesh of the underlying simplicial subdivision goes to zero. The existence of the paths to be described in this section will be proved in section 5. Throughout this paper x_{jh} denotes the h-th component of $x_j \in S^{n_j}$ where $x = (x_1^T,\ldots,x_N^T)^T$ lies in S and $z_{jh}(x)$ denotes the h-th component of $z_j(x) \in R^{n_j+1}$ with $z(x) = (z_1^T(x),\ldots,z_N^T(x))^T$ in $\Pi_{j=1}^{N} R^{n_j+1}$. Finally, $e_j(k)$ is the k-th unit vector in R^{n_j+1}, $k = 1,\ldots, n_j+1$, $j = 1,\ldots,N$.

Let v be an arbitrarily chosen starting point in S. We first describe the sum-process leading from v to a solution point. As its name indicates there are $\Sigma_{j=1}^{N}(n_j+1)$ rays along which the starting point v can be left. The ray along which v is left is uniquely determined by the function value z(v) at v. In the sequel, let I_N be the index set

$\{1,\ldots,N\}$ and for each $j \in I_N$, let $I(j)$ be the set of pairs of indices $\{(j,1),\ldots,(j,n_j+1)\}$ and let $I = \bigcup_{j \in I_N} I(j)$.
The set τ^1 is the collection of subsets of I given by

$$\tau^1 = \{T \subset I \,|\, \text{for all } j \in I_N, \text{ there is a } (j,k) \in I(j)\backslash T_j \text{ with } v_{jk} > 0\},$$

where $T_j = T \cap I(j)$, $j \in I_N$. So, if $T \in \tau^1$, then for all $j \in I_N$ at least one index (j,k) for which v_{jk} is positive does not belong to T. Clearly, the set $T = \{(j,h)\}$ is in τ^1 only if $v_{jh} < 1$. For each $T \in \tau^1$ we define the set $A^1(T)$ by

$$A^1(T) = \{x \in S \,|\, \text{for all } j \in I_N,\ x_{jk} > b_j v_{jk} \text{ if } (j,k) \in T \text{ and}$$
$$x_{jk} = b_j v_{jk} \text{ if } (j,k) \notin T \text{ where } 0 < b_j < 1\}.$$

Notice that

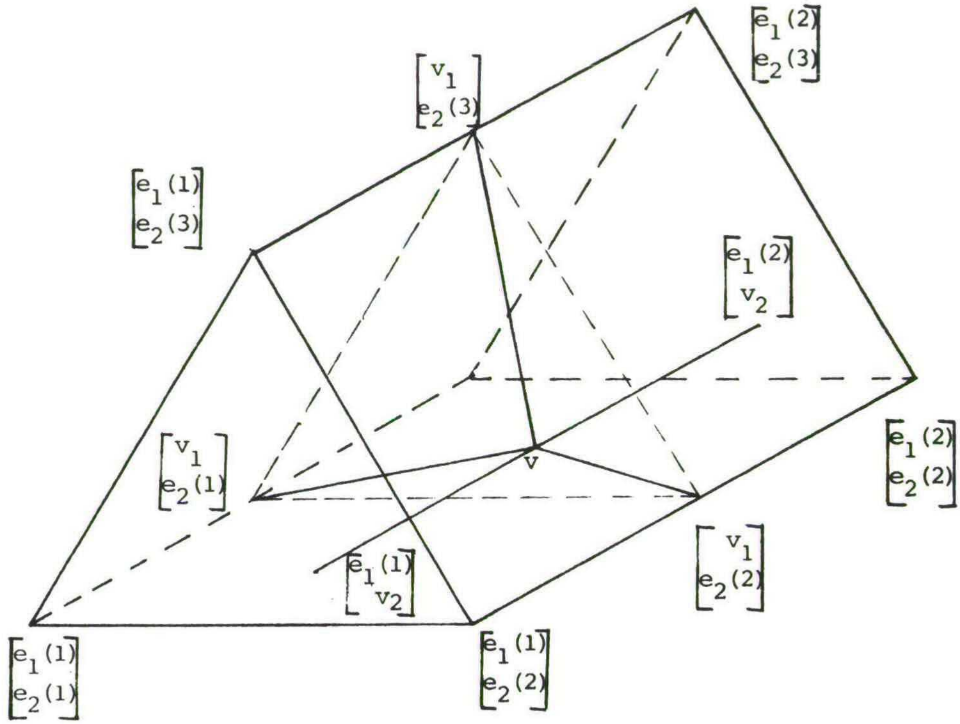
$$A^1(\emptyset) = \{x \in S \,|\, \text{for all } j \in I_N,\ x_{jk} = b_j v_{jk}$$
$$\text{for all } (j,k) \in I(j), \text{ with } 0 < b_j < 1\} = \{v\}.$$

Furthermore, if $T = \{(j,h)\}$, $A^1(T)$ is a one-dimensional subset of S being the line segment

$$A^1(\{(j,h)\}) = \{x \in S \,|\, x_{jh} > b_j v_{jh} \text{ and } x_{jk} = b_j v_{jk},\ k \neq h,$$
$$\text{with } 0 < b_j < 1, \text{ and } x_{ik} = v_{ik}$$
$$\text{for all } i \neq j,\ k = 1,\ldots,n_i+1\}.$$

For simplicity we denote $A^1(\{(j,h)\})$ by $A^1(j,h)$, $(j,h) \in I$. $A^1(j,h)$ is a straight line from $v = (v_1^T, v_2^T, \ldots, v_N^T)^T$ to the point $(v_1^T, v_2^T, \ldots, v_{j-1}^T, e_j^T(h), v_{j+1}^T, \ldots, v_N^T)^T$. There are $\Sigma_{j=1}^N (n_j+1)$ of such rays if all components v_{jh} are less than one, i.e. if, for all j, $v_j \neq e_j(k)$, $k = 1,\ldots,n_j+1$. In general, the number of rays is equal to $\Sigma_{j=1}^N (n_j+1-\ell_j)$, where $\ell_j=1$ if

$v_{jk}=1$ for some $(j,k) \in I(j)$ and $\ell_j=0$ otherwise. For $N=2$, $n_1=1$, $n_2=2$, the rays $A^1(j,h)$, $(j,h) \in I$, are pictured in figure 2.1. In general the dimension of $A^1(T)$ equals $|T|$, with $|T|$ the cardinality of the set T.



<u>Figure 2.1</u>. $N=2$, $n_1=1$, $n_2=2$. The 5 rays from v to $(v_1^T, e_2^T(k))^T$, $k = 1,2,3$, and $(e_1^T(k), v_2^T)^T$, $k = 1,2$.

Besides the sets $A^1(T)$, $T \in \tau^1$, we define for each $T \subset I$ the set $C^1(T)$ by

$$C^1(T) = \{x \in S \mid z_{jk}(x) = \max_{(i,h) \in I} z_{ih}(x), \text{ if } (j,k) \in T\}.$$

In particular, for each $(j,h) \in I$,

$$C^1(j,h) := C^1(\{(j,h)\}) = \{x \in S \mid z_{jh}(x) = \max_{(i,k) \in I} z_{ik}(x)\},$$

and hence $C^1(T) = \bigcap_{(j,h) \in T} C^1(j,h)$. Since each $C^1(j,h)$, $(j,h) \in I$, is closed and since S is covered by all the sets $C^1(j,h)$, it follows from the intersection point theorem on S (see van der Laan, Talman and Van der Heyden [16] or Freund [4]) that there is an x^* in S such that for at least one $j \in I_N$,

$$x^*_{jh} = 0 \text{ or } x^* \in C^1(j,h),\ h = 1,\dots,n_j+1.$$

Since $x_j^T z_j(x) = 0$ for all $x \in S$, it follows that $z_j(x^*) \leq 0$ with $z_{jk}(x^*) = 0$ if $x^*_{jk} > 0$. By definition of the sets $C^1(j,k)$ this implies that $\max_{(i,h)} z_{ih}(x^*) = 0$ and hence $z(x^*) \leq 0$.

The sum-process is now descibed by the sets $B^1(T)$, $T \in \tau^1$, defined by

$$B^1(T) = A^1(T) \cap C^1(T).$$

Under some regularity condition, the union of all sets $B^1(T)$, $T \in \tau^1$, to be denoted by B^1, consists of a disjoint set of piecewise smooth loops and paths in S. In section 5 we will prove that just one of these paths has v as one of its end points, whereas all other end points of the paths are a solution point. So, there is just one path leading from v to a solution point. More precisely, each set $B^1(T)$, $T \in \tau^1$, consists of smooth loops and paths with each path having two end points lying on the boundary of $B^1(T)$. Let x be such an end point. Then three cases can happen. First, we may have that $x_{jk} = 0$ for some $(j,k) \notin T$ with $v_{jk} > 0$. Then, by definition of $A^1(T)$, $b_j = 0$ and hence $x_{jh} = 0$ for all $(j,h) \notin T_j$. Consequently, for $(j,h) \in I(j)$, $x_{jh} > 0$ implies $(j,h) \in T$ and hence for all $(j,h) \in I(j)$ we have that

$$x_{jh} = 0 \text{ or } x \in C^1(j,h)$$

so that x is a solution point. Secondly, we may have that x lies in $A^1(T \setminus \{(j,h)\}) \cap C^1(T)$ for some $(j,h) \in T$. Then x lies also in $B^1(T \setminus \{(j,h)\})$ and is therefore an end point of a path in $B^1(T \setminus \{(j,h)\})$ except when $T = \{(j,h)\}$ in which case $x \in B^1(\emptyset) = \{v\}$. Finally, we may have that x lies in $C^1(T \cup \{(j,k)\}) \cap A^1(T)$ for some $(j,k) \notin T$. Then, either $T \cup \{(j,k)\} \in \tau^1$ and x is also an end point of a path in $B^1(T \cup \{(j,k)\})$, or $T \cup \{(j,k)\} \notin \tau^1$. In the latter case x is a solution point since $T \in \tau^1$ and $T \cup \{(j,k)\} \notin \tau^1$ implies that $v_{jh} = 0$ for all (j,h) not in $T_j \cup \{(j,k)\}$, and hence $x_{jh} = 0$ for all these (j,h).

Linking all paths in $B^1(T)$, $T \in \tau^1$, together, we obtain a collection of loops and paths in B^1. Since v is an end point of a path in $B^1(j,k)$ where $z_{jk}(v) = \max_{(i,h)} z_{ih}(v)$, this collection contains just one path having v as one of its end points. As has been shown above, the other end point must be an intersection point and hence a solution to $z(x) \leq 0$. Furthermore, the end points of all other paths (if any) are solution points. The path connecting v with a solution point x^*, to be denoted by P^1, is the path of points in S generated by the sum-process.

Statement 2.1. The path of points in S followed by the sum-process is the path in B^1 leading from the initial point v to a solution point x^*.

For N=2, $n_1=n_2=1$, the set B^1 and the path P^1 are illustrated in figure 2.2. In this figure $C^1(1,1)$ is denoted by I, $C^1(1,2)$ by II, $C^1(2,1)$ by III and $C^1(2,2)$ by IV.

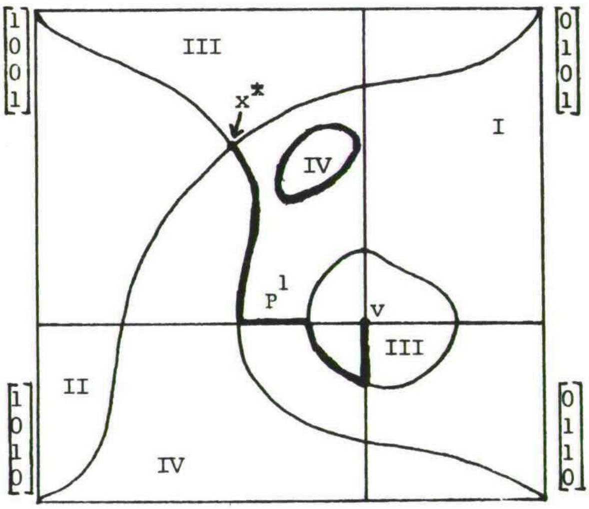


Figure 2.2. B^1 consists of the path P^1 from v to the solution point x^* and a loop in $A^1(\{(1,1),(2,2)\})$.

The product-process is defined in a similar way. Instead of $\Sigma_{j=1}^{N}(n_j+1)$ rays this process is characterized by $\Pi_{j=1}^{N}(n_j+1)$ rays to leave v, each of them leading to a vertex of S. Only if v is a vertex of S this number is one less. Let τ^2 be the collection of subsets of indices given by

$$\tau^2 = \{T \subset I \mid T = \emptyset \text{ or } |T_j| \geq 1,\ j = 1,\dots,N, \text{ and } \exists j \in I_N \text{ for which } \sum_{(j,k) \in T_j} v_{jk} < 1\}.$$

We now define sets $A^2(T)$, $T \in \tau^2$, by

$$A^2(T) = \{x \in S \mid x_{jk} \geq b\, v_{jk} \quad \text{if } (j,k) \in T$$
$$x_{jk} = b\, v_{jk} \quad \text{if } (j,k) \notin T$$
$$\text{where } 0 \leq b \leq 1\}.$$

Clearly, $A^2(\emptyset) = \{v\}$. Furthermore, for all $T \in \tau^2$, $T \neq \emptyset$,

$$\dim A^2(T) = |T| - N + 1.$$

In particular, $A^2(T)$ is a one-dimensional set connecting v with a vertex of S if $|T_j| = 1$ for all $j \in I_N$. This vertex is given by $(e_1^T(k_1),\dots, e_N^T(k_N))^T$ for $T = \{(j,k_j) \in I \mid j = 1,\dots,N\}$. So the number of rays $A^2(T)$ is equal to $\Pi_{j=1}^{N}(n_j+1)$ if v is not a vertex of S and one less if v is a vertex of S. In the latter case, the set of indices corresponding to this vertex does not belong to τ^2 because $v_{jk_j} = 1$ for all j. For N=2, $n_1=1$ and $n_2=2$, the rays are illustrated in figure 2.3.
For $T \subset I$, we define the sets $C^2(T)$ by

$$C^2(T) = \{x \in S \mid z_{jk}(x) = \max_{(j,h) \in I(j)} z_{jh}(x),$$
$$\text{for all } (j,k) \in T_j,\ j = 1,\dots,N\}.$$

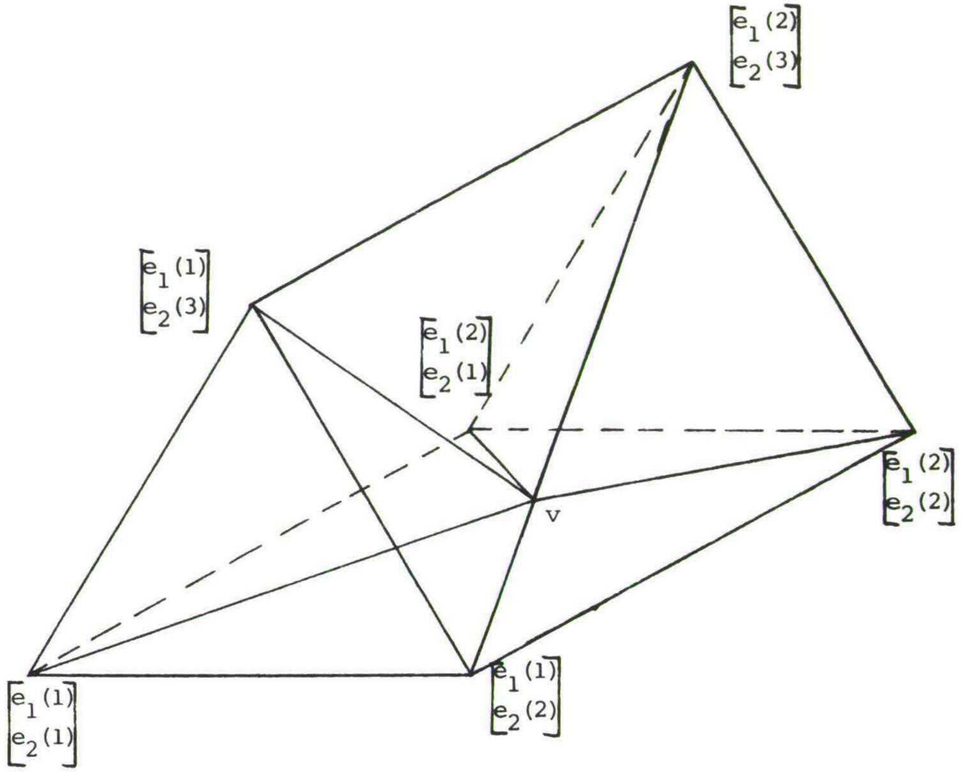


Figure 2.3. The six rays from v to the vertices of S for N=2, n_1=1, n_2=2.

The product-process is now defined by sets $B^2(T)$, $T \in \tau^2$, where for all $T \in \tau^2$,

$$B^2(T) = A^2(T) \cap C^2(T).$$

As will be shown in section 5, the union B^2 of all $B^2(T)$'s, $T \in \tau^2$, consists in general of disjoint loops and paths. One of these paths has v in $B^2(\emptyset) = \{v\}$ as one of its end points, whereas all other end points are a solution point. More precisely, each $B^2(T)$, $T \in \tau^2$, consists of smooth loops and paths with two end points. Each end point x lies in $\mathrm{bd}(A^2(T)) \cap C^2(T)$ or in $A^2(T) \cap C^2(T \cup \{(j,k)\})$, for some $(j,k) \notin T$. If x lies in $\mathrm{bd}(A^2(T))$, then according to the definition of $A^2(T)$, either x lies in $S(T) = \{x \in S | x_{jk} = 0,\ (j,k) \notin T\}$ or x lies in

$A^2(T\setminus\{j,h\})$ for some $(j,h) \in T$, or if $|T_j| = 1$ for all j, $x=v$ and hence an end point of a path in B^2. In the first case, x lies in $S(T) \cap C^2(T)$, so that $z_{ik}(x) = \max_{(i,h) \in I(i)} z_{ih}(x)$ for all (i,k) with $x_{ik} > 0$. Hence, since $x_i^T z_i(x) = 0$, $i \in I_N$, we have that x is a solution point. When x lies in $A^2(T\setminus\{(j,h)\})$ for some $(j,h) \in T$, the point x is also an end point of a path in $B^2(T\setminus\{(j,h)\})$. So, if $x \in \mathrm{bd}(A^2(T)) \cap C^2(T)$, either x is an end point of a path in B^2 or x is an end point of a path in $B^2(T\setminus\{(j,h)\})$ for some $(j,h) \in T$. On the other hand, when an end point x of a path in $B^2(T)$ lies in $C^2(T \cup \{(j,k)\})$ for some $(j,k) \notin T$, x is an end point of a path in $B^2(T \cup \{(j,k)\})$ if $\bar{T} = T \cup \{(j,k)\} \in \tau^2$ or x is a solution point if $\bar{T} \notin \tau^2$. Clearly, $\bar{T} \notin \tau^2$ implies that $v_{jh} = 0$ for all $(j,h) \notin \bar{T}$. So, according to the definition of $A^2(T)$, also $x_{jh} = 0$ for all $(j,h) \notin \bar{T}$, since $T \subset \bar{T}$. Hence, $x_{jk} > 0$ implies $(j,k) \in \bar{T}$ so that $z_{jk}(x) = \max_{(j,h) \in I(j)} z_{jh}(x)$ because $x \in C^2(\bar{T})$. From $x_j^T z_j(x) = 0$ it follows that $z_{jk}(x) = 0$ for $(j,k) \in \bar{T}$ and so $z_{jk}(x) < 0$ for $(j,k) \notin \bar{T}$.

By linking all paths in the various $B^2(T)$'s, $T \in \tau^2$, we obtain the loops and paths of B^2. The starting point v is again an end point of just one path in B^2. Therefore, there exists a path, to be denoted by P^2, in B^2, connecting v with a solution point x^*.

Statement 2.2. The path of the product-process on S is the path P^2 in B^2 which leads from v to a solution point.

For $N=2$, $n_1=n_2=1$, the set B^2 and the path P^2 are illustrated in figure 2.4. In this figure $C^2(\{(1,1),(2,1)\})$ is denoted by I, $C^2(\{(1,1),(2,2)\})$ by II, $C^2(\{(1,2),(2,1)\})$ by III and $C^2(\{(1,2),(2,2)\})$ by IV.

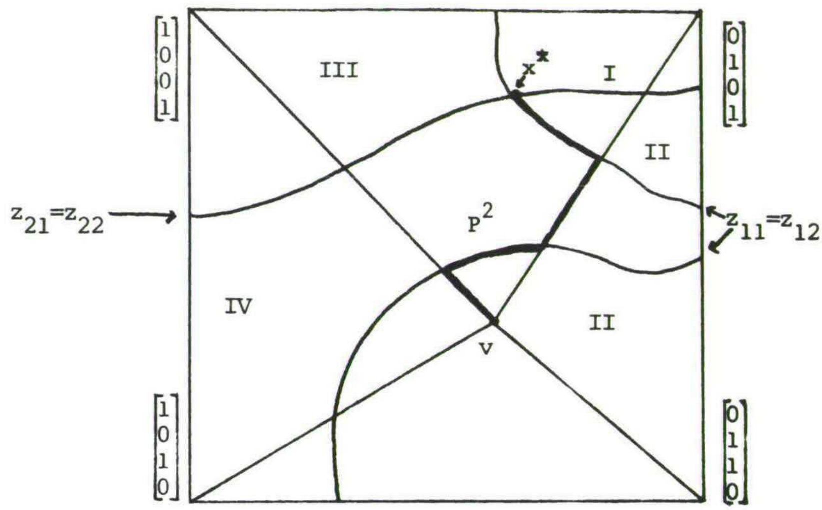


Figure 2.4. B^2 consists of the path P^2 from v to the solution point x^*.

3. The exponent-process on S

Doup, van der Laan and Talman [2] presented a simplicial algorithm on the n-dimensional unit simplex S^n, in which the starting point v can be left along $2^{n+1}-2$ rays. The adjustment process corresponding to this algorithm has been described in van der Laan and Talman [15]. In this section we give a generalization of this adjustment process for problems on S. This new process can leave an interior starting point along $\Pi_{j=1}^{N}(2^{n_j+1}-2)$ rays. We will call it therefore the exponent-process on S. Along which ray the process leaves v depends on the sign pattern of the function value z(v). Since the process is governed by the sign pattern of the function values we introduce some notation on this matter.

A vector $s = (s_1^T, s_2^T, \ldots, s_N^T)^T \in \Pi_{j=1}^{N} R^{n_j+1}$ is a sign vector if $s_{jk} \in \{-1,0,1\}$ for all $(j,k) \in I$. For a sign vector s we define

$$I_j^-(s) = \{(j,k) \in I(j) \,|\, s_{jk} = -1\}$$

$$I_j^0(s) = \{(j,k) \in I(j) \,|\, s_{jk} = 0\}$$

$$I_j^+(s) = \{(j,k) \in I(j) \,|\, s_{jk} = +1\}$$

and

$$I^-(s) = \cup_j I_j^-(s),\ I^0(s) = \cup_j I_j^0(s) \text{ and } I^+(s) = \cup_j I_j^+(s),$$

where the union is over all $j \in I_N$. Furthermore, we denote

$$V_j = \{(j,k) \in I(j) \,|\, v_{jk} = 0\}$$

and $V = \cup_j V_j$. Finally, $h_j = |V_j|$, $V_j^c = I(j) \setminus V_j$, $j = 1,2,\ldots,N$, and $V^c = \cup_j V_j^c$.

Let τ^3 be the set of sign vectors $s \in R^{N+M}$, $M = \sum_{j=1}^{N} n_j$, defined by

$$\tau^3 = \{s \in R^{N+M} \mid \forall j \in I_N, \text{ either } I_j^+(s) = \emptyset \text{ or } I_j^-(s) \cap V_j^c \neq \emptyset,$$

$$\text{and } \exists\, j \in I_N \text{ with } I_j^+(s) \neq \emptyset\}.$$

So, if $s \in \tau^3$, then there is at least one $j \in I_N$ for which there is an index $(j,k) \in I(j)$ with $s_{jk} = +1$ whereas for all $j \in I_N$ we have that if $I_j^+(s)$ is not empty, there is an index $(j,h) \in I(j)$ with $s_{jh} = -1$ and $v_{jh} > 0$ while $I_j^+(s)$ is empty if there is no such index.

As for the sum- and product-process we now define regions in S, which impose conditions on x. For $s \in \tau^3$, the region $A^3(s)$ in S is defined by

$$A^3(s) = \{x \in S \mid \begin{array}{ll} x_{jk} = (1+\lambda_j)v_{jk} & \text{if } s_{jk} = 1 \text{ and } v_{jk} > 0 \\ x_{jk} = \lambda_j & \text{if } s_{jk} = 1 \text{ and } v_{jk} = 0 \\ x_{jk} = b\, v_{jk} & \text{if } s_{jk} = -1 \\ b\, v_{jk} < x_{jk} < (1+\lambda_j)v_{jk} & \text{if } s_{jk} = 0 \text{ and } v_{jk} > 0 \\ x_{jk} < \lambda_j & \text{if } s_{jk} = 0 \text{ and } v_{jk} = 0 \\ \text{with, for all } j,\ \lambda_j > 0, \text{ and } 0 < b < 1\}. & \end{array}$$

Notice that $b = \min_{(j,k) \in V^c} x_{jk}/v_{jk}$ and that for $j = 1,\dots,N$,

$$\lambda_j = \max\{\max_{(j,k) \in V_j^c} x_{jk}/v_{jk},\ 1 + \max_{(j,k) \in V_j} x_{jk}\} - 1$$

if $s_{jk} = 1$ for at least one $(j,k) \in I(j)$. We define $\lambda_j = +\infty$ if $s_{jk} < 0$ for all $(j,k) \in I(j)$.

Clearly, the dimension of $A^3(s)$ is equal to $1 + \Sigma_{j=1}^N |I_j^0(s)| = 1 + |I^0(s)|$ if $I_j^-(s) \cap V_j^c \neq \emptyset$ for all j. If, for some j, $I_j^-(s) \subset V_j$ and so $I_j^+(s) = \emptyset$, we have that for all x in $A^3(s)$

$$\sum_{(j,k) \in I_j^0(s)} x_{jk} = \sum_{(j,k) \in I_j^0(s)} v_{jk}$$

so that the dimension decreases with one for each j for which $I_j^-(s) \subset V_j$.

So,

$$\dim A^3(s) = 1 + \Sigma_{j=1}^N (|I_j^0(s)| - k_j(s))$$

with $k_j(s) = 1$ if $I_j^-(s) \subset V_j$ and $k_j(s) = 0$ otherwise. Observe that $k_j(s) = 1$ implies that both $I_j^-(s) \subset V_j$ and $I_j^+(s) = \emptyset$ and therefore that $I_j^0(s) \neq \emptyset$. From this, it follows that $\dim A^3(s) = 1$ if for all j, $|I_j^0(s)| - k_j(s) = 0$. So, $\dim A^3(s) = 1$ if for all j, either $k_j(s) = |I_j^0(s)| = 0$ and hence $I_j^0(s) = \emptyset$, or $k_j(s) = |I_j^0(s)| = 1$, while for at least one j, $I_j^0(s) = \emptyset$. Since $k_j(s) = |I_j^0(s)| = 1$ implies $I_j^-(s) \subset V_j$, $I_j^+(s) = \emptyset$ and $|I_j^0(s)| = 1$, this can only occur if for some $(j,h) \in I(j)$, $v_{jh} = 1$ and $I_j^-(s) = \{(j,k) \in I(j) | k \neq h\} = V_j$. The set of sign vectors $s \in \tau^3$ which induce a one-dimensional region $A^3(s)$ is therefore given by

$$\tau^3(v) = \{s \in \tau^3 | \text{for all } j \in I_N, \text{ either } I_j^0(s) = \emptyset \text{ or for some } (j,h) \in I(j),\ v_{jh} = 1,\ s_{jh} = 0 \text{ and } I_j^-(s) = V_j \text{ while } \exists j \in I_N \text{ with } I_j^0(s) = \emptyset\}.$$

For $s \in \tau^3(v)$ the one-dimensional region $A^3(s)$ is the line segment connecting $v = (v_1^T, v_2^T, \ldots, v_N^T)^T \in S$ and the point $x = (x_1^T, x_2^T, \ldots, x_N^T)^T$ on bd S, where, if $I_j^0(s) = \emptyset$,

$$x_{jk} = (1+\bar{\lambda}_j)v_{jk} \qquad \text{if } s_{jk} = 1 \text{ and } v_{jk} > 0$$

$$x_{jk} = \bar{\lambda}_j \qquad \text{if } s_{jk} = 1 \text{ and } v_{jk} = 0$$

$$x_{jk} = 0 \qquad \text{if } s_{jk} = -1$$

with $\bar{\lambda}_j$ such that $\Sigma_k x_{jk} = \sum_{(j,k) \in V_j^c \cap I_j^+(s)} (1+\bar{\lambda}_j)v_{jk} + \sum_{(j,k) \in V_j \cap I_j^+(s)} \bar{\lambda}_j = 1$ and where $x_j = v_j$ if $I_j^0(s) \neq \emptyset$, $j \in I_N$. We can consider the point x as the (relative) projection of the point v on the boundary set $S(s) = \{x \in S | x_{jk} = 0$ for all (j,k) with $s_{jk} = -1\}$ of S.

The number of one-dimensional sets or rays follows from comparing τ^3 and $\tau^3(v)$. If v is an interior point of S, and hence $v_{jk} > 0$ for all $(j,k) \in I$, each sign vector s with $I_j^0(s) = \emptyset$, $|I_j^+(s)| \geq 1$ and $|I_j^-(s)| \geq 1$ for all $j \in I_N$, gives a ray. So, in this case there are $\Pi_{j=1}^N(2^{n_j+1}-2)$ rays.

If v is on the boundary there are several cases to consider. First, suppose that for all j, $v_{jh} > 0$ for at least two indices $(j,h) \in I(j)$. Then, again we have that $I_j^0(s) = \emptyset$ while both $I_j^+(s)$ and $I_j^-(s)$ contain at least one element. Moreover $I_j^+(s) \neq \emptyset$ implies $I_j^-(s) \cap V_j^c \neq \emptyset$ and hence all sign vectors s with $s_{jh} = 1$ for all (j,h) with $v_{jh} > 0$ are not allowed. It follows that there are $2^{n_j+1} - 2 - (2^{h_j}-1) = 2^{n_j+1} - 2^{h_j} - 1$ (with $h_j = |V_j|$) possibilities to choose s_j and therefore the total number of rays in this case is $\Pi_{j=1}^N(2^{n_j+1}-2^{h_j}-1)$.

Secondly, suppose that there are at most N-1 indices j with $v_{jh} = 1$ for some $(j,h) \in I(j)$. Then the same holds as above, except that for such an index j also the vector s_j with $s_{jh} = 0$ and $s_{jk} = -1$, $k \neq h$, is allowed, so that the number of rays is $\Pi_{j=1}^N(2^{n_j+1}-2^{h_j}-i_j)$ with $i_j = 0$ if $h_j = n_j$ and $i_j = 1$ if $h_j < n_j$.

Finally, we consider the case that v is a vertex of S, i.e. $h_j = n_j$ for all j. Now, there are again $2^{n_j+1} - 2^{n_j} = 2^{n_j}$ possibilities to choose s_j. However, according to the definition of the set τ^3, $I_j^+(s)$ must be nonempty for at least one j so that the sign vector s for which $s_{jh} = 0$ for all (j,h) with $v_{jh} = 1$ and $s_{ik} = -1$ for all other (i,k) must be excluded.

Combining all the cases above we obtain that the number of one-dimensional sets $A^3(s)$ is equal to

$$\Pi_{j=1}^{N}(2^{n_j+1}-2^{h_j}-i_j) - i_0$$

with $i_0 = 1$ if $h_j = n_j$ for all j and $i_0 = 0$ otherwise.

The sets $A^3(s)$ are illustrated in the figures 3.1 and 3.2 for N=2, $n_1=n_2=1$, and N=2, $n_1=1$, $n_2=2$, respectively.

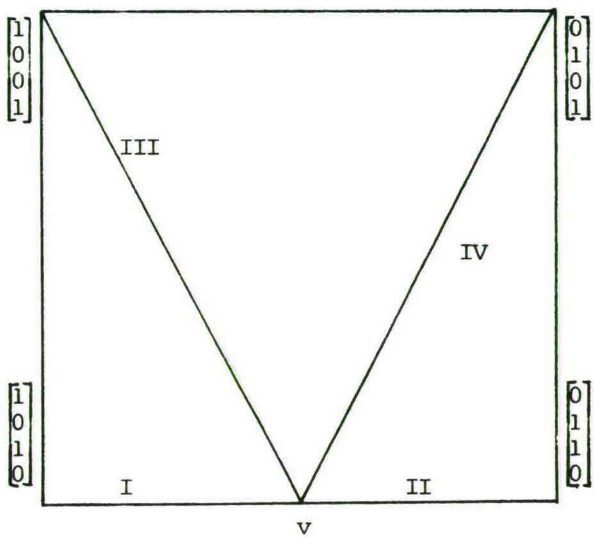


Figure 3.1. The four rays $A^3(s)$ leaving v for $s = (1,-1,0,-1)^T$, $s = (-1,1,0,-1)^T$, $s = (1,-1,-1,1)^T$ and $s = (-1,1,-1,1)^T$ denoted by I, II, III, and IV respectively; $v = (\frac{1}{2},\frac{1}{2},1,0)^T$.

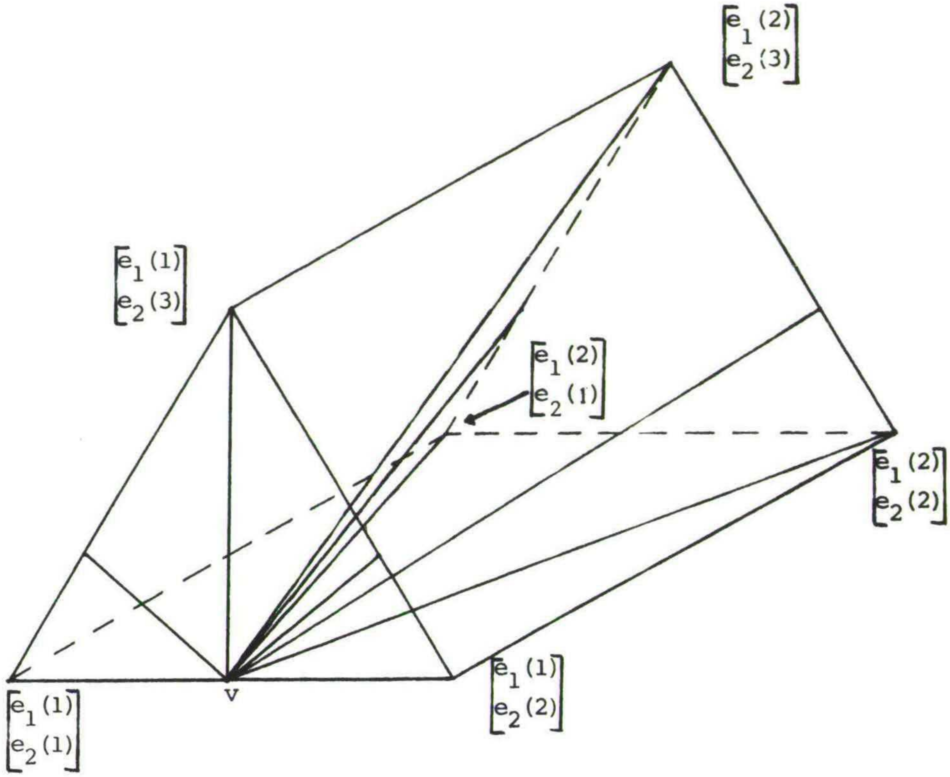


Figure 3.2. The ten rays leaving v; $v_{12} = v_{23} = 0$.

Next we define the regions in S which impose conditions on the function values. For each sign vector s, the subset $C^3(s)$ of S is defined by

$$C^3(s) = C\ell\{x \in S \,|\, \text{sign } z(x) = s\},$$

where $C\ell(W)$ denotes the closure of the set W. Clearly $C^3(s) = \emptyset$ if for some j, $s_{jh} = 1$ for all $(j,h) \in I(j)$ or $s_{jh} = -1$ for all $(j,h) \in I(j)$, since $x_j^T z_j(x) = 0$ for all $j \in I_N$ and $x \in S$. On the other hand, $z(x) < 0$ if $x \in C^3(s)$ with $s_{jh} \in \{-1,0\}$ for all $(j,h) \in I$ and hence each x in such a subset $C^3(s)$ is a solution point. Notice that for a solution point x in $C^3(s)$, $s_{jh} = -1$ implies $x_{jh} = 0$.

We now consider the sets $B^3(s)$, $s \in \tau^3$, defined by

$$B^3(s) = A^3(s) \cap C^3(s).$$

For simplicity we assume that v is not a solution point so that $s^0 = \operatorname{sign} z(v) \nleq 0$ and $s^0 \in \tau^3$. If for some j, $v_{jh} = 1$ and $z_{jk}(v) > 0$ for at least one $k \neq h$, we set $s^0_{jh} = -1$, although $z_{jh}(v) = 0$. Clearly $v \in B^3(s^0)$. Again, under some regularity conditions, the union B^3 of all sets $B^3(s)$, $s \in \tau^3$, consists of a disjoint collection of paths and loops, containing one path with v as one of its end points and a solution point as its other end point, whereas all other paths have two solution points as their end points. This will be proved in section 6. To be more explaining, each set $B^3(s)$, $s \in \tau^3$, consists of smooth loops and paths. An end point x of a path lies either in bd $A^3(s)$ or in bd $C^3(s)$. We first consider the case that an end point x lies in bd $A^3(s)$. We say that s^1 conforms to s^2, denoted by $s^1 \leftarrow s^2$ if $s^1_{jh} \neq 0$ implies $s^2_{jh} = s^1_{jh}$. Moreover, we say that s^1 conforms closely to s^2, $s^1 \twoheadleftarrow s^2$ if $s^1 \leftarrow s^2$ and $\dim A^3(s^2) = \dim A^3(s^1) - 1$. Since $\dim A^3(s) = 1 + \Sigma^N_{j=1}(|I^0_j(s)| - k_j(s))$, $s^1 \twoheadleftarrow s^2$ implies that for just one j, $s^2_{jh} \in \{-1,+1\}$ for just one $(j,h) \in I^0_j(s^1)$ if $k_j(s^1) = 0$. In case $k_j(s^1) = 1$, either for just two indices $(j,h) \in I^0_j(s^1)$ and $(j,k) \in I^0_j(s^1) \cap V^c_j$, $s^2_{jh} = 1$ and $s^2_{jk} = -1$ or $s^2_{jh} = -1$ for just one $(j,h) \in V_j$ for which $s^1_{jh} = 0$.

Clearly,

$$\text{bd } A^3(s) = [S(s) \cap A^3(s)] \cup [\cup\{A^3(s^1) \mid s \twoheadleftarrow s^1\}].$$

<u>Lemma 3.1</u>. If, for some $s \in \tau^3$, $x \in B^3(s) \cap S(s)$, then x is a solution point.

<u>Proof</u>. Since $x \in S(s)$ we have that $x_{jk} = 0$ for all $(j,k) \in I^-(s)$. Consequently, $z_{jk}(x) = 0$ for all (j,k) with $x_{jk} > 0$. Suppose now that for some (j,k) with $x_{jk} = 0$, $z_{jk}(x) > 0$. Since $z_{jk}(x) > 0$ we have that $(j,k) \in I^+(s)$. However, $x_{jk} = 0$ and $x \in A^3(s)$ implies both $v_{jk} = 0$ and $\lambda_j = 0$ and therefore b=1. Hence x=v. However $s \in \tau^3$ implies that $v \notin S(s)$, which proves that $z(x) \leq 0$. □

So, if an end point x of a path in $B^3(s)$ lies in $S(s)$ then x is a solution point. On the other hand, if $x \in A^3(s^1)$ with $s \dashv s^1$ then x is an end point of a path in $B^3(s^1)$ since $x \in C^3(s^1)$.

We now consider the case that $x \in \text{bd } C^3(s)$. Then, $x \in C^3(s^1)$, either for some $s^1 \in \tau^3$ with $s^1 \dashv s$ or for some $s^1 \notin \tau^3$. In the first case also $x \in A^3(s^1)$ and hence x is an end point of a path in $B^3(s^1)$. If, however, $s^1 \notin \tau^3$, then we have that x is a solution point.

Lemma 3.2. If $x \in C^3(s)$ for some $s \notin \tau^3$, then $z(x) < 0$.

Proof. Because $s \notin \tau^3$, either for all j, $I_j^+(s) = \emptyset$ or there is a j with $I_j^+(s) \neq \emptyset$ and $I_j^-(s) \cap V_j^c = \emptyset$. In the first case we have that $s_{jk} \in \{-1,0\}$ for all (j,k) and hence $s < 0$, implying that $z(x) < 0$. In the latter case $s_{jh} \in \{1,0\}$ for all $(j,h) \in I(j)$ with $x_{jh} > 0$, which implies $z_{jh}(x) = 0$ for all (j,h) with $x_{jh} > 0$ because $x_j^T z_j(x) = 0$. Hence $z_{jh}(x) > 0$ implies $x_{jh} = 0$. However, as in the proof of lemma 3.1 this implies $v_{jh} = 0$ and $\lambda_j = 0$ and therefore $x=v$. However, v lies in $C^3(s^0)$ and $s^0 \in \tau^3$. □

Concluding we have that an end point of a path in $B^3(s)$ is either an end point of a path in $B^3(s^1)$ with $s^1 \dashv s$ or $s \dashv s^1$, or is a solution point, or is v itself, being the end point of a path in $B^3(s^0)$. By linking all paths in $B^3(s)$, $s \in \tau^3$, we obtain the collection B^3 of loops and paths. The starting point v is an end point of just one path, say P^3, in B^3. The other end point of this path must be a solution point.

Statement 3.3. The path of the exponent-process on S is the path P^3 in B^3 which leads from v to a solution point.

Observe that the product- and exponent-process coincide when $n_j=1$ for all j. So, for an illustration we refer to figure 2.4 where $C^3((1,-1,1,-1)^T)$ is denoted by I, $C^3((1,-1,-1,1)^T)$ by II, $C^3((-1,1,1,-1)^T)$ by III, and $C^3((-1,1,-1,1)^T)$ by IV.

4. The sum-, product- and exponent-process as adjustment processes

The paths traced by the processes given in the previous sections can be seen as paths of points x in S along which x is adjusted to reach a solution point. Since for each process the path starting in v reaches a solution point x^* we have convergent adjustment processes. In case of a noncooperative N-person game the variable x_{jh} denotes the probability with which player j plays his h-th strategy and we may speak about strategy adjustment processes. In economic modelling a variable denotes a price (or some other economic variable) and we have price adjustment processes. The three processes differ in the way in which the variables are adjusted. However, in all processes the components of $z(x)$ satisfy certain conditions along the trajectory.

In the sum-process the starting point v, when not a solution point, is left along the ray $A^1(j,h)$ with (j,h) the index of the component of $z(v)$ with the highest value. We assume that this index is unique. Along this ray the variable x_{jh} is increased, while all other components x_{jk}, $k \neq h$, of x_j are equally decreased relatively to v_{jk}, until $z_{ik}(x)$ becomes equal to $z_{jh}(x)$ for some index $(i,k) \neq (j,h)$. Then the process continues in $A^1(T)$ with $T = \{(j,h),(i,k)\}$, tracing a path of points x along which $z_{ik}(x)$ is equal to $z_{jh}(x)$ but larger than the values of the other components of $z(x)$. Continuing the process, a path of points x is traced along which, for varying T, $T \in \tau^1$, x lies in $A^1(T) \cap C^1(T)$, so that for all $j \in I_N$ the ratio between the components x_{jh}, $(j,h) \notin T_j$, is equal to the ratio of these components at v but smaller than the ratio between a component x_{jk}, $(j,k) \in T_j$, and a component x_{jh}, $(j,h) \notin T_j$. For the indices $(j,k) \in T_j$ the value $z_{jk}(x)$ is kept equal to the maximum of $z_{ih}(x)$ over all components $(i,h) \in I$. When the path reaches a part of bd S from which the process didn't start, i.e. x_{jh} becomes zero for some $(j,h) \notin T_j$ with $v_{jh} > 0$, then a solution point is found. If for some $(j,k) \in T_j$ and $(j,h) \notin T_j$, the ratio x_{jk}/x_{jh} becomes equal to v_{jk}/v_{jh}, so that x lies in $A^1(T\setminus\{(j,k)\})$, then this ratio is not further decreased but is kept equal to v_{jk}/v_{jh} and $z_{jk}(x)$ is decreased away from $\max_{(i,h)} z_{ih}(x)$. On the other hand, if for some $(j,k) \notin T$, $z_{jk}(x)$ becomes equal to the maximum of the components of $z(x)$, then either x is a solution point if $T \cup \{(j,k)\} \notin \tau^1$, or the process is continued in $B^1(T \cup \{(j,k)\})$ by increasing the ratio between x_{jk} and the components x_{jh} of x with (j,h) not in T_j while $z_{jk}(x)$ is kept equal to the maximum of the components of z(x). The trajectory of this adjustment process converges to a solution point x^*.

In the sum-process initially only the component (j,h) of x corresponding to the component of z with the highest value is increased. So, initially the variables x_{ik}, $i \neq j$, are not adjusted. This is not the case in the product-process. Assuming that v itself is not a solution point and that each set I(j) contains a unique index (j,k_j) for which $\max_{(j,k) \in I(j)} z_{jk}(v)$ is attained, this process leaves the initial point v along the ray $A^2(T)$, where $T = \{(j,k_j) \in I \mid j = 1,\ldots,N\}$. Along this ray, the components x_{jk_j}, $j = 1,\ldots,N$, are increased, while all other components of x are equally decreased relatively to the components of v, until for some (j,k), $k \neq k_j$, $z_{jk}(x)$ becomes equal to $z_{jk_j}(x)$. Then the process continues in $A^2(T \cup \{(j,k)\})$, keeping z_{jk} equal to z_{jk_j} but larger than the other components of z_j. More generally, for varying T, $T \in \tau^2$, the product-process traces a path of points x in $A^2(T) \cap C^2(T)$, i.e., the ratio between two components x_{jk} and x_{ih}, (j,k) and (i,h) both not in T, is kept equal to the ratio v_{jh}/v_{ih} and these ratios are smaller than the ratio between x_{jk}, $(j,k) \in T$, and x_{ih}, $(i,h) \notin T$, while for the indices $(j,k) \in T_j$, $z_{jk}(x)$ is kept equal to the maximum of $z_{jh}(x)$ over the indices $(j,h) \in I(j)$, $j = 1,\ldots,N$. Notice the difference between the product- and the sum-process. In the latter process the maximum is taken over all components of z, whereas in the product-process, the maximum is taken over the components of z_j, $j \in I_N$. On the other hand, in the sum-process $x_{jk} = b_j v_{jk}$ if $(j,k) \notin T$, whereas in the product-process $x_{jk} = bv_{jk}$, i.e., in the latter process the ratio between the components of x with indices not in T is equal to the same ratio at v, whereas in the sum-process this only holds for ratios between two components with indices both in I(j) for some $j \in I_N$. The process stops in a solution point whenever it reaches a part of bd S from which it didn't start, i.e. x_{jk} becomes zero for all $(j,k) \notin T$. When in the product-process the ratio between two components with indices (j,k) in T and (i,h) not in T becomes equal to v_{jk}/v_{ih}, then (j,k) is deleted from T and the process continues by decreasing $z_{jk}(x)$ away from $\max_{(j,h) \in I(j)} z_{jh}(x)$ and keeping x_{jk}/x_{ih} equal to v_{jk}/v_{ih}. If for some (j,k) not in T, $z_{jk}(x)$ becomes equal to the maximum of $z_{jh}(x)$ over all $(j,h) \in I(j)$, then either a solution point is found if $T \cup \{(j,k)\}$ is not in τ^2 or the process continues in $B^2(T \cup \{(j,k)\})$ by increasing the ratio between x_{jk} and x_{ih} with (i,h) not in T while $z_{jk}(x)$ is kept equal to the maximum of $z_{jh}(x)$ over all components (j,h) in I(j). Again, the trajectory converges to a solution point x^*.

Finally, we consider the exponent-process. Recall that the product-process initially adjusts x by only increasing the variables x_{jk} with indices (j,k) for which $z_{jk}(x)$ has the highest value over all components of $z_j(x)$. In the exponent-process, all variables are adjusted simultaneously. The starting point v is left along the ray $A^3(s^0)$ with $s^0 = \operatorname{sign} z(v)$. Along this ray the variables x_{jk} with indices (j,k) for which $z_{jk}(x)$ is positive are increased, keeping, for all j, the ratio between two of these variables with indices in I(j) equal to their ratio at v, with some modifications when $v_{jk} = 0$. Furthermore, the variables x_{jk} with indices (j,k) for which $z_{jk}(x)$ is negative are all proportionally decreased. The process follows this ray until a solution point is found or $z_{jk}(x)$ becomes zero for some index $(j,k) \in I$. In the latter case $z_{jk}(x)$ is kept equal to zero and x_{jk} is varied between the relative lower bound bv_{jk} and the upper bound (if any) $(1+\lambda_j)v_{jk}$ when $v_{jk} > 0$ or λ_j when $v_{jk} = 0$. Here $b = x_{i\ell}/v_{i\ell}$ for the indices (i,ℓ) for which $z_{i\ell}(x) < 0$ and $v_{i\ell} > 0$ while $x_{jh} = (1+\lambda_j)v_{jh}$ (or λ_j) for the indices (j,h) for which $z_{jh}(x) > 0$ and $v_{jh} > 0$ ($v_{jh} = 0$). In general, for varying sign vectors s, $s \in \tau^3$, the process generates a path of points x in $A^3(s) \cap C^3(s)$. So, $z_{jk}(x) = 0$ and x_{jk} lies between the relative bounds bv_{jk} and $(1+\lambda_j)v_{jk}$ (or λ_j when $v_{jk} = 0$) if $s_{jk} = 0$, $z_{jk}(x) > 0$ and $x_{jk} = (1+\lambda_j)v_{jk}$ (or λ_j when $v_{jk} = 0$) if $s_{jk} > 0$, while $z_{jk}(x) < 0$ and $x_{jk} = bv_{jk}$ if $s_{jk} < 0$. As soon as $z_{jk}(x)$ becomes equal to zero while $I_j^+(s) = \{(j,k)\}$ and $I_j^-(s) \cap V_j^c = \{(j,h)\}$ then also $z_{jh}(x)$ becomes zero by the complementarity conditions on z. The process continues in $B^3(s^1)$ where $s^1_{jk} = 0$, $s^1_{jh} = 0$ and $s^1_{ip} = s_{ip}$ for all $(i,p) \neq (j,k), (j,h)$, keeping z_{jk} and z_{jh} equal to zero. If $z_{jk}(x)$ becomes equal to zero for some $(j,k) \in I_j^+(s)$ with $| I_j^+(s) | > 1$ or $(j,k) \in I_j^-(s) \cap V_j$ or $(j,k) \in I_j^-(s) \cap V_j^c$ with $| I_j^-(s) \cap V_j^c | > 1$ then the process continues in $A^3(s^1)$ with $s_{jk}^1 = 0$ and $s_{\ell p}^1 = s_{\ell p}$ for all other indices $(\ell,p) \in I$ while z_{jk} is kept equal to zero. However, s_{jk} changes from zero to 1 or -1 when x_{jk} reaches its relative upper- or lower-bound. When $I_j^+(s) = \emptyset$ and $I_j^-(s) \subset V_j$ there exists a unique pair of indices (j,k), (j,h) both in $I_j^0(s)$ with $v_{jh} > 0$ for which x_{jk} is on its upper-bound and x_{jh} reaches its lower-bound. While keeping x_{jk} and x_{jh} equal to these bounds, the process increases $z_{jk}(x)$ and decreases $z_{jh}(x)$ away from zero. When $I_j^+(s) \neq \emptyset$ only one element s_{jk} becomes 1 or -1 and z_{jk} is increased or decreased away from zero respectively depending on whether x_{jk} reaches its relative upper- or lower-bound.

The process stops whenever it reaches a part of bd S from which it didn't start or an area $C^3(s)$ with $s \notin \tau^3$. Again the adjustment process induced by the exponent-process converges to a solution point x^* as will be shown in section 6.

5. Existence proofs I

In the sections 2 and 3 we described the so-called sum-, product-, and exponent-process on the product space of several unit simplices, the simplotope S. However, the existence proofs were postponed. In this and the next section we show that under some very weak conditions there indeed exist paths leading from the starting point v to a solution point x^* of the underlying problem.

Central in the argumentation will be the concept of a primal-dual pair of subdivided manifolds - abbreviated PDM - which has been introduced in Kojima and Yamamoto [7]. After a short description of some notions and properties we present in this section for the sum- and the product-process the appropriate PDM from which the existence of the path is obtained. The exponent-process will be considered in the next section.

By a cell in R^k we mean a convex polyhedral set being the intersection of a finite number of closed half spaces. If a cell D is a face of a cell E, we write $D < E$. Let $\mathcal{M}$ be a finite collection of m-dimensional cells or m-cells in R^k. We denote the collection of faces $\{D \mid D < E, E \in \mathcal{M}\}$ of $\mathcal{M}$ by $\bar{\mathcal{M}}$ and the union of all m-cells E, $E \in \mathcal{M}$, by $|\mathcal{M}|$. We call $\mathcal{M}$ a subdivided manifold if

a) for all D, $E \in \mathcal{M}$, $D \cap E = \emptyset$ or $D \cap E$ is a common face of both D and E

b) each (m-1)-cell in $\bar{\mathcal{M}}$ lies in at most two m-cells of $\mathcal{M}$

c) $\mathcal{M}$ is locally finite, i.e. each point x in $|\mathcal{M}|$ has a neighbourhood which intersects with only a finite number of m-cells in $\mathcal{M}$.

The boundary of $\mathcal{M}$, denoted by $\delta\mathcal{M}$, is the collection of all (m-1)-cells of $\bar{\mathcal{M}}$ which lie in only one m-cell of $\mathcal{M}$.

Definition 5.1. Let m be a positive integer. We say that the triple $(\mathcal{P}, \mathcal{D}, d)$ is a PDM with degree m if it satisfies the following conditions

1) P and D are subdivided manifolds with dual operator d;

2) $\forall X \in \bar{P}$: $X^d \in \bar{D}$ or $X^d = \emptyset$;

$\forall Y \in \bar{D}$: $Y^d \in \bar{P}$ or $Y^d = \emptyset$;

3) $Z \in \bar{P} \cup \bar{D}$ and $Z^d \neq \emptyset$ implies $(Z^d)^d = Z$;

4) If $X_1, X_2 \in \bar{P}$, $X_1 < X_2$, $X_1^d \neq \emptyset$ and $X_2^d \neq \emptyset$, then $X_2^d < X_1^d$;

If $Y_1, Y_2 \in \bar{D}$, $Y_1 < Y_2$, $Y_1^d \neq \emptyset$ and $Y_2^d \neq \emptyset$, then $Y_2^d < Y_1^d$;

5) If $Z \in \bar{P} \cup \bar{D}$ and $Z^d \neq \emptyset$, then $\dim Z + \dim Z^d = m$.

P (D) is said to be the primal (dual) subdivided manifold, and $|P|$ ($|D|$) the primal (dual) manifold. We call Z^d the dual of Z for every Z in $\bar{P} \cup \bar{D}$. Next, we define

$$L = \langle P, D, d\rangle = \{Y^d \times Y \mid Y \in \bar{D} \text{ and } Y^d \neq \emptyset\}.$$

Observe that $\langle P, D, d\rangle = \{X \times X^d \mid X \in \bar{P} \text{ and } X^d \neq \emptyset\}$. If (P, D, d) is a PDM with degree m, then $L = \langle P, D, d\rangle$ is a subdivided m-manifold. Moreover, suppose that $D = X \times Y$ is an (m-1)-cell of $\bar{L}$, where $X \in \bar{P}$ and $Y \in \bar{D}$. Then if E is an m-cell of L having D as one of its faces $E = X \times X^d$ or $E = Y^d \times Y$. For the proofs and more details we refer to Kojima and Yamamoto [7].

Now, let M be a subdivided (n+1)-manifold in R^k and let h be a piecewise continuously differentiable (PC^1) mapping from $|M|$ into R^n, i.e. the restriction of h to each (n+1)-cell C of M can be extended to a continuously differentiable mapping on an open neighbourhood of C. We call c in R^n a regular value of h if $\dim h(B) = n$ for all cells B in $\bar{M}$ for which $c \in h(B)$. If c is a regular value of h then the set

$$h^{-1}(c) = \{x \in |M| \mid h(x) = c\}$$

does not intersect with any face B in $\bar{M}$ of dimension less than n.

<u>Theorem 5.2.</u> Let M be a subdivided (n+1)-manifold in R^k and let $h : |M| \to R^n$ be a PC^1 mapping. Suppose that $c \in R^n$ is a regular value of h. Then $h^{-1}(c)$ is a disjoint union of paths and loops satisfying the following properties:

i) if $E \in \mathcal{M}$ and $h^{-1}(c) \cap E \neq \emptyset$, then $h^{-1}(c) \cap E$ is a disjoint union of smooth 1-manifolds;

ii) each loop has no intersection with $|\delta \mathcal{M}|$;

iii) $x \in h^{-1}(c)$ is an end point of a path if and only if $x \in |\delta \mathcal{M}|$;

iv) if $|\mathcal{M}|$ is a closed subset of R^k every open or semiclosed path is unbounded.

Proof. See Eaves [3].

Corollary 5.3. Let $\mathcal{L}$ be a subdivided n-manifold induced by a PDM with degree n and let R_+ denote the set of nonnegative real numbers, then

$$\mathcal{K} = \{Z \times R_+ \mid Z \in \mathcal{L}\} \tag{5.1}$$

is an (n+1)-manifold with boundary equal to

$$\delta \mathcal{K} = \{Z \times \{0\} \mid Z \in \mathcal{L}\} \cup \{Z' \times R_+ \setminus \{0\} \mid Z' \in \delta \mathcal{L}\}.$$

Moreover, if h is a PC^1-mapping from $|\mathcal{K}|$ to R^n and 0 is a regular value of h, then $(x,t) \in h^{-1}(0)$ is an end point of a path if and only if $t=0$ (and $x \in \text{int}\,|\mathcal{L}|$) or $x \in |\delta \mathcal{L}|$ (and $t > 0$).

To prove the existence and convergence of the paths of the three processes on S we will define suitable PDM's and PC^1 mappings h on the corresponding manifold as defined in (5.1) and apply theorem 5.2 to deduce that $h^{-1}(0)$ contains a path which leads from v to a solution point.

Starting with the sum-process, the primal is completely determined by the sets $A^1(T)$, $T \in \tau^1$, and the dual is induced by the sets $C^1(T)$, $T \subset I$. More precisely, for $T \subset I$ let the set $Y^1(T)$ be given by

$$Y^1(T) = \{y \in R^{N+M} \mid y_{jk} = \max_{(i,h) \in I} y_{ih} = 1 \text{ for all } (j,k) \in T\}$$

Moreover, we define $Y_0^1 = \{y \in R^{N+M} \mid y_{jk} < 1 \text{ for all } (j,k) \in I\} = Y^1(\emptyset)$ and $A_0^1 = \{v\} = A^1(\emptyset)$.

<u>Theorem 5.4</u>. The triplet $(\mathcal{P}^1, \mathcal{D}^1, d^1)$ is a PDM with degree N+M where

a) $\mathcal{P}^1 = \{A^1(T) \mid T \in \tau^1 \text{ and } |T| = M\}$
$\bar{\mathcal{P}}^1 = \{A^1(T) \mid T \in \tau^1\} \cup \{A^1_j(T) \mid T \in \tau^1 \text{ and } j \in I_N\}$ with
for $j \in I_N$ and $T \in \tau^1$

$$A^1_j(T) = \{x \in A^1(T) \mid x_{jk} = 0,\ (j,k) \in I(j) \setminus T_j\}$$

b) $\mathcal{D}^1 = \{Y^1_0\}$
$\bar{\mathcal{D}}^1 = \{Y^1(T) \mid T \subset I\}$

c) $(A^1(T))^{d^1} = Y^1(T)$ for all $A^1(T) \in \bar{\mathcal{P}}^1$
$(A^1_j(T))^{d^1} = \emptyset$ for all $T \in \tau^1$ and $j \in I_N$
$(Y^1(T))^{d^1} = A^1(T)$ for all $T \in \tau^1$
$(Y^1(T))^{d^1} = \emptyset$ for all $T \subset I$ and $T \notin \tau^1$.

In particular $(A^1_0)^{d^1} = Y^1_0$ and $(Y^1_0)^{d^1} = A^1_0$.

<u>Proof</u>. From the definition of the $A^1(T)$'s we see that the boundary of $A^1(T)$, $T \in \tau^1$, is equal to

$$\text{bd } A^1(T) = \Big(\bigcup_{(i,h) \in T} A^1(T \setminus \{(i,h)\})\Big) \cup \Big(\bigcup_{j \in I_N} A^1_j(T)\Big). \quad (5.2)$$

Since each $A^1(T)$, $T \in \tau^1$, is a $|T|$-cell and $A^1_j(T)$ with $T \in \tau^1$ and $j \in I_N$, is a $(|T|-1)$-cell in δS, the collection $\mathcal{P}^1$ is a subdivided M-manifold with $\bar{\mathcal{P}}^1$ as stated under a. Furthermore, each $Y^1(T)$, $T \subset I$, is an $(M+N-|T|)$-cell with boundary equal to

$$\text{bd } Y^1(T) = \bigcup_{(i,h) \in I \setminus T} Y^1(T \cup \{(i,h)\})$$

so that $\mathcal{D}^1$ is a subdivided (M+N)-manifold with $\bar{\mathcal{D}}^1 = \{Y^1(T) \mid T \subset I\}$. Finally, the triplet $(\mathcal{P}^1, \mathcal{D}^1, d^1)$ by definition satisfies the other conditions of definition 5.1. In particular, for all $T \in \tau^1$

$$\dim A^1(T) + \dim Y^1(T) = M + N. \qquad \square$$

We call $(\mathcal{P}^1, \mathcal{D}^1, d^1)$ the PDM with respect to the sum-process. Now, let $\mathcal{L}^1$ be the (M+N)-manifold $\langle \mathcal{P}^1, \mathcal{D}^1, d^1 \rangle$ consisting of the (M+N)-cells $A^1(T) \times Y^1(T)$, $T \in \tau^1$. Then we define the (M+N+1)-manifold $\mathcal{K}^1$ by

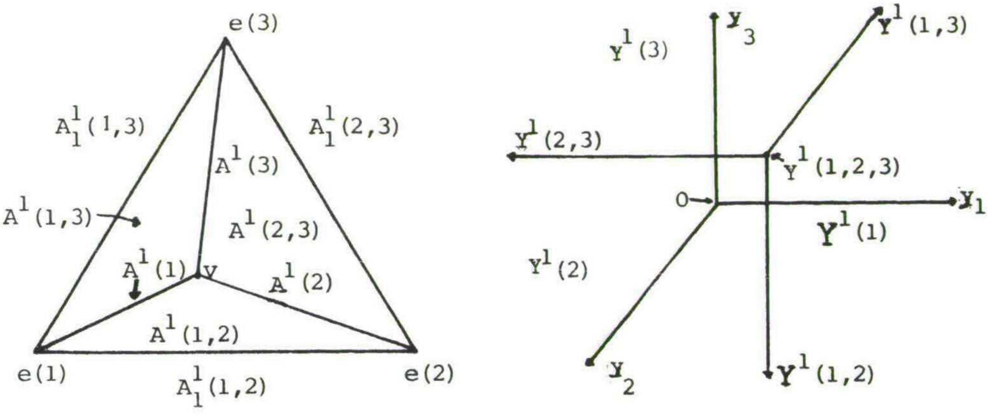
$$\mathcal{K}^1 = \{Z \times R_+ \mid Z \in \mathcal{L}^1\}.$$

From corollary 5.3 and (5.2) we obtain immediately that the boundary of $\mathcal{K}^1$ is equal to

$$|\delta \mathcal{K}^1| = \Big(\bigcup_{\substack{T \in \tau^1 \\ T \cup \{(i,h)\} \notin \tau^1}} (A^1(T) \times Y^1(T \cup \{(i,h)\}) \times R_+ \setminus \{0\}) \Big) \cup$$

$$\Big(\bigcup_{\substack{T \in \tau^1 \\ j \in I_N}} (A^1_j(T) \times Y^1(T) \times R_+ \setminus \{0\}) \Big) \cup \Big(\bigcup_{T \in \tau^1} (A^1(T) \times Y^1(T) \times \{0\}) \Big) \qquad (5.3)$$

Observe that $A^1(T) \times Y^1(T \cup \{(i,h)\}) \times R_+ \setminus \{0\}$ is indeed a facet of two (M+N+1)-cells if $T \cup \{(i,h)\} \in \tau^1$. The PDM is illustrated in figure 5.1 for the case N=1, n_1=2, where the index (1,k) is denoted by k, k = 1,2,3.



a. The primal P^1, $A_0^1 = \{v\}$

b. The dual D^1,
$Y_0^1 = \{y \in R^3 \mid y_j < 1,\ j \in I_3\}$

Figure 5.1. The PDM with respect to the sum-process.

Notice that the point (v,0,0) lies in $|\delta K^1|$ since the zero-vector lies in Y_0^1 and v lies in A_0^1. Finally, we define the function $h^1 : |K^1| \to R^{N+M}$ by

$$h^1(x,y,t) = y - tz(x) \quad , \quad (x,y,t) \in |K^1| \tag{5.4}$$

Assumption 5.5. The function $z : S \to R^{N+M}$ is a continuously differentiable mapping.

Under assumption 5.5 the mapping h^1 is a piecewise continuously differentiable mapping from $|K^1|$ to R^{N+M}.

Assumption 5.6. The point 0 in R^{N+M} is a regular value of the mapping h^1.

From (5.3) and theorem 5.2 we then immediately obtain that under the assumptions 5.5 and 5.6 the point (v,0,0) is an end point of a path in $(h^1)^{-1}(0)$ and that the latter set consists of a disjoint union of piecewise smooth loops and paths, each path having 0, 1 or 2 end points in $|\delta K^1|$.

Lemma 5.7. The point $(v,0,0)$ is the only end point of a path in $(h^1)^{-1}(0)$.

Proof. Suppose that (x,y,t) is an end point of a path in $(h^1)^{-1}(0)$. According to theorem 5.2 the point (x,y,t) must lie in $|\delta K^1|$. When $t=0$, we get from (5.4) $y=0$ and so $x=v$ since $0=y \in \text{int } Y_0^1$ and $(Y_0^1)^{d^1} = A_0^1 = \{v\}$. So, let t be positive. Then from (5.3) we obtain that there must be a $T \in \tau^1$ such that

$$(x,y,t) \in A^1(T) \times Y^1(T \cup \{(i,h)\}) \times R_+ \backslash \{0\}$$

for some index (i,h) with $T \cup \{(i,h)\}$ not in τ^1, or

$$(x,y,t) \in A_j^1(T) \times Y^1(T) \times R_+ \backslash \{0\} \qquad \text{for some } j \in I_N.$$

Suppose that the first case holds. Since $T \cup \{(i,h)\} \notin \tau^1$ we have $\Sigma_k v_{ik} = 1$ where the sum is over all indices (i,k) in $T_i \cup \{(i,h)\}$. So, since $x \in A^1(T)$,

$$x_{ik} = 0 \qquad \text{for all } (i,k) \notin T_i \cup \{(i,h)\}. \tag{5.5}$$

On the other hand, y lies in $Y^1(T \cup \{(i,h)\})$, i.e., for all (i,k) in $T_i \cup \{(i,h)\}$,

$$z_{ik}(x) = y_{ik}/t = 1/t$$

so that for all these indices $z_{ik}(x) > 0$. Hence, from (5.5) it follows that

$$x_i^T z_i(x) = \sum_{T_i \cup \{(i,h)\}} x_{ik} z_{ik}(x) > 0$$

which contradicts the fact that $x_i^T z_i(x) = 0$. The second case is similar since $x \in A_j^1(T)$ for some $j \in I_N$ implies $\Sigma_{T_j} x_{jk} = 1$ whereas $tz_{jk}(x) = y_{jk} = 1$ for all $(j,k) \in T_j$ so that $x_j^T z_j(x) = \Sigma_{T_j} x_{jk} z_{jk}(x) > 0$.

□

Lemma 5.7 shows that the piecewise smooth path G^1 in $(h^1)^{-1}(0)$ with end point $(v,0,0)$ is a semiclosed and unbounded path whereas all other paths in $(h^1)^{-1}(0)$ have no end points at all and are both open and unbounded. Observe that if $z(v) < 0$, then $G^1 = \{(x,y,t) \in |K^1| \mid x = v, y = tz(v), t > 0\}$. We assume in the following that v does not solve the NLCP on S with respect to z.

Since P^1 is a bounded M-manifold, at least one of the y_{jk}'s goes to minus infinity or t goes to plus infinity on the unbounded path G^1 in $(h^1)^{-1}(0)$ originating in $(v,0,0)$. Suppose that one of the y_{jk}'s goes to $-\infty$. Since $y_{jk} = tz_{jk}(x)$ on the path and $z_{jk}(x)$ is bounded on the compact S we must have also that t goes to infinity. However, $h^1(x,y,t) = 0$ implies

$$z_{jk}(x) = y_{jk}/t \leq 1/t \qquad \text{for all } (j,k) \in I$$

with an equality for at least one index (if $x \neq v$). Therefore, when t goes to infinity on an unbounded path in $(h^1)^{-1}(0)$ we must have that all components of $z(x)$ tend to be nonpositive. More precisely, the unbounded path G^1 originating in $(v,0,0)$ must approach a limit point $(x^*,y^*) \in |L^1|$ with $z(x^*) \leq 0$. Observe that on the path for a point (x,y,t),

$$t = \Big(\max_{(i,h) \in I} z_{ih}(x)\Big)^{-1} \text{ and } y = z(x)/\max_{(i,h) \in I} z_{ih}(x)$$

holds. The next theorem says that the set B^1 defined in section 2 is the projection of $(h^1)^{-1}(0)$ on $|P^1|$ and that the path P^1 of the sum-process is the projection of the piecewise smooth path G^1 on $|P^1|$.

Theorem 5.8. Let (x,y,t) be a point in $(h^1)^{-1}(0)$. Then there is a $T \in \tau^1$ such that

$$x \in B^1(T) = A^1(T) \cap C^1(T).$$

Proof. The proof of lemma 5.7 shows that if (x,y,t) belongs to $(h^1)^{-1}(0)$ and $t=0$, then $(x,y,t) = (v,0,0)$. So, suppose that $(x,y,t) \in (h^1)^{-1}(0)$ with $t > 0$. For t, $0 < t < (\max_I z_{ih}(v))^{-1}$, the point (v,y,t) lies in $(h^1)^{-1}(0)$ with

$$y = tz(v) \in \text{int } Y_0^1$$

so that (v,y,t) also lies in G^1 and $v \in B^1(\emptyset)$.
When $t = (\max_I z_{ih}(v))^{-1}$ and $x=v$, the point (x,y,t) lies in G^1 with

$$y = z(v)/\max_I z_{ih}(v) \in \text{bd } Y_0^1.$$

More precisely, $y \in Y^1(\{(j,k)\})$ with (j,k) the unique index for which

$$z_{jk}(v) = \max_{(i,h) \in I} z_{ih}(v).$$

Therefore, $v \in C^1(j,k)$. Since v also lies in $A^1(j,k)$, we have that

$$v \in B^1(\{(j,k)\}).$$

In general, if (x,y,t) belongs to $(h^1)^{-1}(0)$ and $x \neq v$, there is an index set $T \in \tau^1$ such that

$$(x,y,t) \in A^1(T) \times Y^1(T) \times R_+\backslash\{0\}$$

whereas

$$y - z(x)t = 0.$$

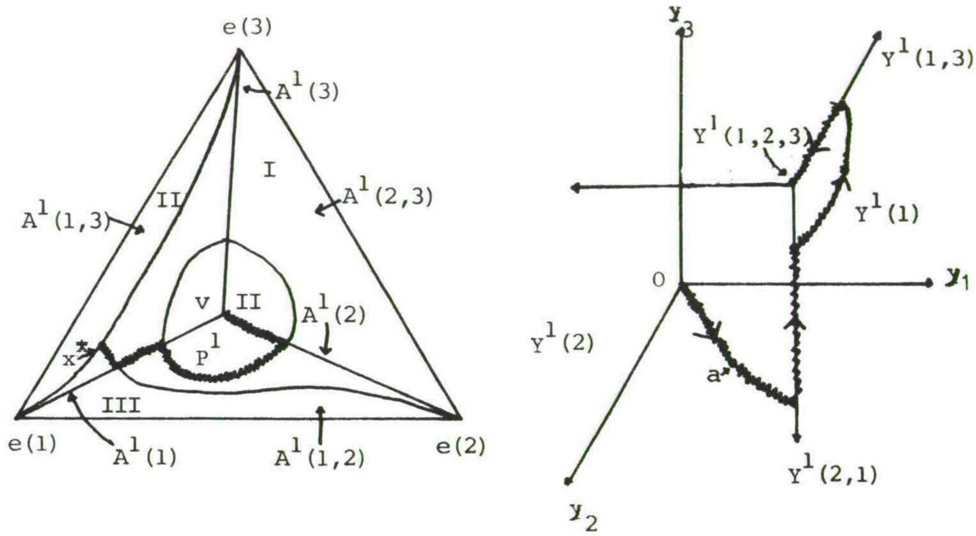
So, $y = tz(x)$ and $y \in Y^1(T)$, i.e.

$$z_{jk}(x) = \max_{(i,h) \in I} z_{ih}(x) = 1/t > 0 \qquad \text{for all } (j,k) \in T.$$

Recall that $y_{jk} = tz_{jk}(x) = 1$, $(j,k) \in T$. Consequently, x lies in both $A^1(T)$ and $C^1(T)$, which completes the proof of the theorem. □

Corollary 5.9. Under the assumptions 5.5 and 5.6 the path P^1 of the sum-process exists and connects the starting point v with a solution to the NLCP on S with respect to z.

The path G^1 projected on $|P^1|$ and $|D^1|$ respectively is illustrated in figure 5.2 for N=1 and $n_1=2$. Again, the index (1,k) is denoted by k, k = 1,2,3.



a. Projection P^1 of G^1 on $|P^1|$ b. Projection of G^1 on $|D^1|$

Figure 5.2. The projections of G^1 on $|P^1|$ and $|D^1|$ for N=1, $n_1=2$. The projections are heavily drawn. In v, $z_2(v) = \max_i z_i(v)$ and $a = z(v)/z_2(v)$ lies in $Y^1(\{2\})$.
G^1 lies in $A^1(T) \times Y^1(T) \times R_+\backslash\{0\}$ for subsequently $T = \emptyset$, $\{2\}$, $\{2,1\}$, $\{1\}$ and $\{1,3\}$. G^1 approaches the point (x^*,y^*) when t goes to infinity with x^* in $A^1(\{1,3\})$ and $\{y^*\} = Y^1(\{1,2,3\})$. $C^1(\{1\})$ is denoted by I, $C^1(\{2\})$ by II and $C^1(\{3\})$ by III.

We remark that if 0 is not a regular value of h^1 we can replace the system of equations $h^1(x,y,t) = 0$ by $h^1(x,y,t) = c$ where c is a regular value arbitrarily close to 0. The initial point in $|\delta K^1|$ then becomes $(v,c,0)$ with $c \in \operatorname{int} Y^1_0$. Secondly, if we consider the sum-ray algorithm on S which focusses on the minimum, the limiting path also fits in this framework. Finally we remark that for both sum-ray algorithms the parameter t does not need to go monotonically from 0 to infinity on the path G^1.

Next we examine the existence and convergence of the path $\mathcal{P}^2$ of the product-process on S. Since most of the arguments are the same as for the sum-process we confine ourselves to a short description. We start with defining the appropriate PDM. Let I^2 be the collection of proper index sets T in I such that $|T_j| > 1$ for all $j \in I_N$. The primal of the PDM with respect to the product-process is completely determined by the $A^2(T)$'s with T in τ^2. The dual is again induced by the $C^2(T)$'s. More precisely, for $T \in I^2$, we define the set $Y^2(T)$ by

$$Y^2(T) = \{y \in R^{M+N} \mid \sum_{j=1}^{N} \max_{(j,k) \in I(j)} y_{jk} = 1 \text{ and}$$

$$y_{jh} = \max_{(j,k) \in I(j)} y_{jk},\ (j,h) \in T\}.$$

Furthermore we define $A_0^2 = \{v\}$ and

$$Y_0^2 = \{y \in R^{M+N} \mid \sum_{j=1}^{N} \max_{(j,k) \in I(j)} y_{jk} < 1\}.$$

Observe that $\dim Y^2(T) = M + 2N - |T| - 1$ and that $\dim Y_0^2 = M + N$.

Theorem 5.10. The triplet $(\mathcal{P}^2, \mathcal{D}^2, d^2)$ is a PDM with degree M+N, where

a) $\mathcal{P}^2 = \{A^2(T) \mid T \in \tau^2 \text{ and } |T| = M + N - 1\}$
$\bar{\mathcal{P}}^2 = \{A^2(T) \mid T \in \tau^2\} \cup A_0^2 \cup \{S(T) \mid T \in \tau^2\}$ with
$S(T) = \{x \in S \mid x_{jk} = 0, \text{ for all } (j,k) \notin T\}$

b) $\mathcal{D}^2 = \{Y_0^2\}$
$\bar{\mathcal{D}}^2 = \{Y^2(T) \mid T \in I^2\} \cup Y_0^2$

c) $(A^2(T))^{d^2} = Y^2(T)$ for all $T \in \tau^2$

$(S(T))^{d^2} = \emptyset$ for all $T \in \tau^2$

$(Y^2(T))^{d^2} = A^2(T)$ for all $T \in \tau^2$

$(Y^2(T))^{d^2} = \emptyset$ for all $T \in I^2 \setminus \tau^2$

$(A_0^2)^{d^2} = Y_0^2$ and $(Y_0^2)^{d^2} = A_0^2$.

Proof. From the definition of the $A^2(T)$'s, $T \in \tau^2$, we obtain that

$$\mathrm{bd}\, A^2(T) = S(T) \cup \Big(\bigcup_{T \setminus \{(i,h)\} \in \tau^2} A^2(T \setminus \{(i,h)\}) \Big) \tag{5.6}$$

if $|T_j| > 1$ for at least one $j \in I_N$. Furthermore, if $|T_j| = 1$ for all $j \in I_N$,

$$\mathrm{bd}\, A^2(T) = S(T) \cup A_0^2. \tag{5.7}$$

Finally, for all $T \in I^2$

$$\mathrm{bd}\, Y^2(T) = \bigcup_{T \cup \{(i,h)\} \in I^2} Y^2(T \cup \{(i,h)\}) \tag{5.8}$$

and

$$\mathrm{bd}\, Y_0^2 = \bigcup_{\{T \,|\, |T_j| = 1 \text{ for all } j \in I_N\}} Y^2(T). \tag{5.9}$$

Both P^2 and D^2 are subdivided manifolds since each $A^2(T)$, $T \in \tau^2$, and $Y^2(T)$, $T \in I^2$, as well as A_0^2 and Y_0^2 is a convex polyhedron. Because of (5.6)-(5.9) and since

$$\dim A^2(T) + \dim Y^2(T) = M + N \quad , \; T \in \tau^2$$

and

$$\dim A_0^2 + \dim Y_0^2 = M + N ,$$

P^2 and D^2 also satisfy the other conditions of definition 5.1. Therefore (P^2, D^2, d^2) is indeed a PDM with degree M+N. □

We call $(\mathcal{P}^2, \mathcal{D}^2, d^2)$ the PDM with respect to the product-process. Now, let $\mathcal{L}^2$ be the (M+N)-manifold $\langle \mathcal{P}^2, \mathcal{D}^2, d^2 \rangle$ and let the (M+N+1)-manifold $\mathcal{K}^2$ be defined by

$$\mathcal{K}^2 = \{X \times R_+ \mid X \in \mathcal{L}^2\}.$$

From the proof of theorem 5.10 we obtain that

$$\begin{aligned}
|\delta\, \mathcal{K}^2| = (& \bigcup_{\substack{T \in \tau^2 \\ T \cup \{(i,h)\} \in I^2 \setminus \tau^2}} (A^2(T) \times Y^2(T \cup \{(i,h)\}) \times R_+ \setminus \{0\})) \cup \\
& (\bigcup_{\substack{T \notin \tau^2 \\ |T_j| = 1, \forall_j}} (A_0^2 \times Y^2(T) \times R_+ \setminus \{0\})) \cup \\
& (\bigcup_{T \in \tau^2} (S(T) \times Y^2(T) \times R_+ \setminus \{0\})) \cup \\
& (\bigcup_{T \in \tau^2} (A^2(T) \times Y^2(T) \times \{0\})) \cup \\
& (A_0^2 \times Y_0^2 \times \{0\}). \qquad (5.10)
\end{aligned}$$

Notice that the point (v,0,0) lies in $|\delta\, \mathcal{K}^2|$ since 0 lies in Y_0^2. The mapping h^2 whose zero points will induce the path of the product-process is given by

$$h^2(x,y,t) = y - tz(x) \qquad ,(x,y,t) \in |\mathcal{K}^2|.$$

<u>Assumption 5.11</u>. The value 0 in R^{N+M} is a regular value of h^2.

Together with assumption 5.5 we have that the function h^2 is a PC^1 mapping on $|\mathcal{K}^2|$ so that according to theorem 5.2, $(h^2)^{-1}(0)$ consists of a disjoint union of loops and paths, each path having 0, 1 or 2 end points in $|\delta\, \mathcal{K}^2|$. Since (v,0,0) lies in $|\delta\, \mathcal{K}^2|$ and $h^2(v,0,0) = 0$, the point (v,0,0) is an end point of a path in $(h^2)^{-1}(0)$.

Lemma 5.12. The point $(v,0,0)$ is the only end point of a path in $(h^2)^{-1}(0)$.

Proof. Let (x,y,t) be an end point of a path in $(h^2)^{-1}(0)$. Then (x,y,t) lies in $|\delta K^2|$. If $t=0$ then $y = 0 \in \text{int } Y_0^2$ so that $x=v$. Suppose now that $t > 0$. Then according to (5.10) either for some $T \in \tau^2$

$$(x,y,t) \in A^2(T) \times Y^2(T \cup \{(i,h)\}) \times R_+\setminus\{0\} \text{ with } T \cup \{(i,h)\} \notin \tau^2$$

or

$$(x,y,t) \in S(T) \times Y^2(T) \times R_+\setminus\{0\},$$

or for some $T \notin \tau^2$ with $|T_j| = 1$ for all $j \in I_N$

$$(x,y,t) \in A_0^2 \times Y^2(T) \times R_+\setminus\{0\}.$$

In the last case we must have $x=v$ and $v_{jk_j} = 1$ for the unique index (j,k_j) in T_j, $j \in I_N$. Since $y \in Y^2(T)$ we must have $\Sigma_{j=1}^N y_{jk_j} = 1$ so that $z_{jk_j}(v) = \max_h z_{jh}(v) > 0$ for at least one index $j \in I_N$. Therefore, for this j

$$v_j^T z_j(v) = v_{jk_j} z_{jk_j}(v) > 0$$

contradicting the fact that $x_{j'}^T z_{j'}(x) = 0$ for all $x \in S$, $j' \in I_N$. Now suppose that for some $T \in \tau^2$ the point (x,y,t) lies in $A^2(T) \times Y^2(T \cup \{(i,h)\}) \times R_+\setminus\{0\}$ for some index (i,h) with $T \cup \{(i,h)\}$ not in τ^2. With $T' = T \cup \{(i,h)\}$, then according to the definitions of τ^2 and $A^2(T)$ for all $j \in I_N$

$$\sum_{(j,k) \in T'_j} x_{jk} = 1.$$

Since $y \in Y^2(T')$ there is an index $j \in I_N$ such that $\max_k y_{jk} > 0$. Therefore, $z_{jk}(x) = t^{-1} y_{jk} > 0$ for all $(j,k) \in T'_j$. Notice that $y_{jh} = \max_k y_{jk}$, $(j,h) \in T'_j$. Consequently, for this index j

$$x_j^T z_j(x) = \sum_{(j,h) \in T'_j} x_{jh} z_{jh}(x) > 0$$

yielding again a contradiction. Finally, suppose that for some $T \in \tau^2$ the point (x,y,t) lies in $S(T) \times Y^2(T) \times R_+ \setminus \{0\}$. Then $x_{jk} = 0$ for all $(j,k) \notin T$. Since $y \in Y^2(T)$ there must be an index $j \in I_N$ with $z_{jh}(x) = \max_k z_{jk}(x) = t^{-1}\max_k y_{jk} > 0$ for all $(j,h) \in T_j$ so that again $x_j^T z_j(x) > 0$. □

Lemma 5.12 shows that the piecewise smooth path G^2 in $(h^2)^{-1}(0)$ with end point $(v,0,0)$ is a semi-closed and unbounded path whereas all other paths in $(h^2)^{-1}(0)$ are open and unbounded. If $z(v) \leq 0$, then

$$G^2 = \{(x,y,t) \in |K^2| \mid x = v,\ y = tz(v),\ t \geq 0\}.$$

So, we assume in the following that v does not solve the NLCP. Since $|P^2| = S$ is bounded at least one of the y_{jk}'s goes to minus infinity or t goes to infinity on the unbounded path G^2 originating in $(v,0,0)$. If one of the y_{jk}'s goes to minus infinity then t goes to infinity since $y_{jk} = tz_{jk}(x)$ on the path and z is bounded on S. However, $h^2(x,y,t) = 0$ implies

$$z_{jk}(x) = y_{jk}/t \leq 1/t \qquad \text{for all } (j,k) \in I$$

with $t = (\Sigma_{j=1}^{N} \max_{(j,k) \in I(j)} z_{jk}(x))^{-1}$. Therefore, when t goes to infinity on an unbounded path in $(h^2)^{-1}(0)$, all components of $z(x)$ tend to be nonpositive so that the path G^2 approaches a limit point $(x^*,y^*) \in |L^2|$ satisfying $z(x^*) \leq 0$. The next corollary concludes that the path P^2 of the product-process is the projection of the path G^2 on $|P^2|$ and that B^2 is the same projection of $(h^2)^{-1}(0)$.

Corollary 5.13. If $(x,y,t) \in (h^2)^{-1}(0)$ then there is a $T \in \tau^2$ such that $x \in B^2(T) = A^2(T) \cap C^2(T)$. More precisely, $(v,y,t) \in G^2$ for all t with

$$0 \leq t \leq (\sum_{j \in I_N} \max_k z_{jk}(v))^{-1} \text{ and } y = tz(v) \in Y_0^2 .$$

When $t = (\Sigma_j \max_k z_{jk}(v))^{-1}$ then $y \in Y^2(T^0)$ where T_j^0 consists of the (unique) index (j,k) for which

$$z_{jk}(v) = \max_{(j,h) \in I(j)} z_{jh}(v) \quad , \; j \in I_N$$

so that v lies in $B^2(T^0)$. If $x \neq v$, then there is a $T \in \tau^2$ such that

$$(x,y,t) \in A^2(T) \times Y^2(T) \times R_+ \setminus \{0\}$$

so that x lies in $B^2(T)$, $t = (\Sigma_j \max_k z_{jk}(x))^{-1}$ and $y = tz(x)$. Moreover, P^2 is the projection of the path G^2 on $|P^2| = S$, P^2 exists and connects v with a solution to the NLCP when the assumptions 5.5 and 5.11 are satisfied.

6. Existence proofs II

Finally, we prove the existence and convergence of the path P^3 belonging to the exponent-process on S. This proof is slightly different from those for the other processes. When treating these processes we used a single PDM with a primal manifold corresponding to the A(T)'s, while the dual was induced by the C(T)'s. The sum of the dimensions of an A(T) and its corresponding C(T) was always constant. Also here, the primal of a suitable PDM is related to the $A^3(s)$-areas and the dual is induced by the $C^3(s)$-areas. However, the sum of the dimensions of $A^3(s)$ and its corresponding dual area isn't constant. The dimension of a cell in the dual can jump with two whereas the dimension of an $A^3(s)$ can only change with one. A jump of two happens when for some $j \in I_N$, $z_j(x)$ becomes equal to or lower than zero or when such a situation is left. Therefore we define a collection of PDM's, each PDM related to a single sign vector $s \in \tau^3$.

First we need to generalize the notion of a regular value. Again we consider a subdivided (n+1)-manifold M and a PC^1 mapping h from $|M|$ into R^n. We call c in R^n a regular value of h if dim h(B) = n for all cells B in M for which $c \in h(B)$ but we allow dim h(C) = n-1 for all (n-1)-cells C being a face of just one (n+1)-cell in M. It follows that for a regular value $c \in R^n$ the set

$$h^{-1}(c) = \{x \in |M| \;|h(x) = c\}$$

only intersects with a face B in $\bar{M}$ of dimension less than n, if B is an (n-1)-face in $|\delta M|$ of just one (n+1)-cell in M.

As said before, we will define a collection of PDM's. The primal of each PDM is determined by a region $A^3(s)$, whereas the dual is induced by the corresponding $C^3(s)$. More precisely, for a sign vector $s \in I^3$ we define the set $Y^3(s)$ by

$$Y^3(s) = C\ell\{y \in R^{N+M} | \text{sign}\, y = s \text{ and } \sum_{(j,k)\in I} y^+_{jk} = 1\}$$

where

$$I^3 = \{s \in R^{N+M} | s \text{ is a sign vector with } |I^+(s)| > 1 \text{ and } |I^-(s)| > 1\}$$

and y^+_{jk} denotes the maximum of zero and y_{jk}.

Theorem 6.1. For $s \in \tau^3$ the triplet $(P^3(s), D^3(s), d^3(s))$ is a PDM of degree $N + M - \Sigma_{j=1}^N k_j(s)$ where

a) $P^3(s) = \{A^3(s)\}$

b) $D^3(s) = \{Y^3(s)\}$

c) $(A^3(s))^{d^3(s)} = Y^3(s)$ and $(Y^3(s))^{d^3(s)} = A^3(s)$ and $X^{d^3(s)} = \emptyset$ for each proper face X of $A^3(s)$ or $Y^3(s)$.

Proof. For each $s \in \tau^3$ the set $A^3(s)$ is a $(\Sigma_{j=1}^N(|I_j^0(s)| - k_j(s)) + 1)$-cell and $Y^3(s)$ is an $(N + M - \Sigma_{j=1}^N |I_j^0(s)| - 1)$-cell, so that both $P^3(s)$ and $D^3(s)$ are subdivided manifolds consisting of one cell. Moreover, since the dual operator transforms only $A^3(s)$ and $Y^3(s)$ into a nonempty set, all the conditions of definition 5.1 are immediately satisfied. Therefore, $(P^3(s), D^3(s), d^3(s))$ is a PDM with degree equal to

$$\dim A^3(s) + \dim Y^3(s) = N + M - \sum_{j=1}^{N} k_j(s). \qquad \square$$

We call the collection $\{(P^3(s), D^3(s), d^3(s)) \mid s \in \tau^3\}$ the set of PDM's with respect to the exponent-process.

For simplicity of notation we define for each $s \in \tau^3$ some related sets of sign vectors corresponding to the boundary of $A^3(s)$ and $Y^3(s)$. Related to the boundary of $A^3(s)$ we define the set P(s) by

$$P(s) = \{\tilde{s} \in \tau^3 \mid s \dashv \tilde{s}\}$$

where $s \dashv \tilde{s}$ means that s conforms closely to $\tilde{s}$ (see section 3).
The set P(s) is subdivided in two sets $P_1(s)$ and $P_2(s)$ defined by

$$P_1(s) = \{\tilde{s} \in \tau^3 \mid s \dashv \tilde{s} \text{ and for just one j, either } \tilde{s}_{jh} \in \{-1,+1\}$$
$$\text{for just one } (j,h) \in I_j^0(s) \text{ if } k_j(s) = 0 \text{ or } \tilde{s}_{jh} = -1$$
$$\text{for just one } (j,h) \in V_j \text{ if } k_j(s) = 1 \text{ and } s_{jh} = 0\}$$

and

$$P_2(s) = \{\tilde{s} \in \tau^3 \mid s \dashv \tilde{s} \text{ and for just one j with } k_j(s) = 1 \text{ and}$$
$$\text{for just two indices } (j,h) \in I_j^0(s) \text{ and } (j,k) \in I_j^0(s) \cap$$
$$V_j^c,\ \tilde{s}_{jh} = 1 \text{ and } \tilde{s}_{jk} = -1\}.$$

Furthermore, let $A_0^3 = \{v\}$.
Corresponding to the boundary of $Y^3(s)$, $s \in \tau^3$, we define the set Q(s) by

$$Q(s) = \{\bar{s} \in I^3 \mid \bar{s}_{ih} = s_{ih} \text{ for all } (i,h) \in I \text{ except for one}$$
$$(j,k) \in I \text{ with } s_{jk} \neq 0 \text{ and } \bar{s}_{jk} = 0\}.$$

Furthermore, recall from section 3 the definition of $\tau^3(v)$, the index set of one-dimensional regions $A^3(s)$, and let $\bar{S}(s)$ be the intersection of $A^3(s)$ and S(s).

Now, let $\mathcal{L}^3(s)$ be the $(N + M - \sum_{j=1}^{N} k_j(s))$-manifold $\langle \mathcal{P}^3(s), \mathcal{D}^3(s), d^3(s)\rangle$ induced by $(\mathcal{P}^3(s), \mathcal{D}^3(s), d^3(s))$.

<u>Theorem 6.2</u>. The boundary of $\mathcal{L}^3(s)$ – $s \in \tau^3$ – is equal to

$$|\partial \mathcal{L}^3(s)| = (\bar{S}(s) \times Y^3(s)) \cup \Big(\bigcup_{\tilde{s} \in P(s)} (A^3(\tilde{s}) \times Y^3(s))\Big) \cup$$
$$\Big(\bigcup_{t \in Q(s)} (A^3(s) \times Y^3(t))\Big) \qquad , s \notin \tau^3(v)$$

and

$$|\partial \mathcal{L}^3(s)| = (\bar{S}(s) \times Y^3(s)) \cup (A_0^3 \times Y^3(s)) \cup (\bigcup_{t \in Q(s)} (A^3(s) \times Y^3(t))) \qquad , s \in \tau^3(v).$$

Proof. It holds that $|\partial \mathcal{L}^3(s)| = (\text{bd } A^3(s) \times Y^3(s)) \cup (A^3(s) \times \text{bd } Y^3(s))$. With $\text{bd } Y^3(s) = \bigcup_{t \in Q(s)} Y^3(t)$, $\text{bd } A^3(s) = A_0^3 \cup \bar{S}(s)$ for $s \in \tau^3(v)$, and $\text{bd } A^3(s) = \bar{S}(s) \cup (\bigcup_{\tilde{s} \in P(s)} A^3(\tilde{s}))$ for all other $s \in \tau^3$, the theorem follows immediately. □

In the next theorem the relation between the different manifolds $\mathcal{L}^3(s)$, $s \in \tau^3$, is stated.

Theorem 6.3. Let $\mathcal{L}^3(s)$ be the manifold related to a sign vector $s \notin \tau^3(v)$. The cell $A^3(\tilde{s}) \times Y^3(s)$ lies in both $\delta\mathcal{L}^3(s)$ and $\delta\mathcal{L}^3(\tilde{s})$ if $\tilde{s} \in P_1(s)$. If $\tilde{s} \in P_2(s)$ the cell $A^3(\tilde{s}) \times Y^3(s)$ lies in $\delta\mathcal{L}^3(s)$ and is a facet of two cells in $\delta\mathcal{L}^3(\tilde{s})$ and therefore a face of $\mathcal{L}^3(\tilde{s})$.
For an arbitrary manifold $\mathcal{L}^3(s)$, $s \in \tau^3$, the cell $A^3(s) \times Y^3(t)$, $t \in Q(s)$, in $\delta\mathcal{L}^3(s)$ lies also in $\delta\mathcal{L}^3(t)$ if $t \in \tau^3$. If $t \notin \tau^3$ and $\exists\ \bar{t} \in \tau^3$ with $\bar{t} \in Q(t)$ and $s \in P_2(\bar{t})$ then $A^3(s) \times Y^3(\bar{t})$ is a facet of two cells in $\delta\mathcal{L}^3(s)$. Moreover, $A^3(s) \times Y^3(\bar{t})$ is a cell in $\delta\mathcal{L}^3(\bar{t})$.

Proof. From theorem 6.2 we deduce that the cell $A^3(\tilde{s}) \times Y^3(s)$, $\tilde{s} \in P_1(s)$, is a facet of both $A^3(s) \times Y^3(s)$ and $A^3(\tilde{s}) \times Y^3(\tilde{s})$. Furthermore, if $\tilde{s} \in P_2(s)$, $A^3(\tilde{s}) \times Y^3(s)$ is a facet of $A^3(s) \times Y^3(s)$ and a facet of both $A^3(\tilde{s}) \times Y^3(t^1)$ and $A^3(\tilde{s}) \times Y^3(t^2)$ with $t^1 \in Q(\tilde{s})$ and $t^2 \in Q(\tilde{s})$. More precisely, if $\tilde{s}_{jk} = 1$ and $\tilde{s}_{jh} = -1$ with $(j,h) \in V_j^c$, $k_j(s) = 1$ and $s_{jh} = s_{jk} = 0$, it holds that $t^1_{jk} = 0$ and $t^2_{jh} = 0$. From theorem 6.2 we know that $A^3(\tilde{s}) \times Y^3(t^1)$ and $A^3(\tilde{s}) \times Y^3(t^2)$ lie in $\delta\mathcal{L}^3(\tilde{s})$.
The proof of the second part of the theorem follows the same lines. □

We now define the $(N + M - \sum_{j=1}^{N} k_j(s) + 1)$-manifold $\mathcal{K}^3(s)$, $s \in \tau^3$, by

$$\mathcal{K}^3(s) = \{X \times R_+ \mid X \in \mathcal{L}^3(s)\}.$$

From theorem 6.2 we derive that for the boundary of $K^3(s)$ holds

$$|\delta K^3(s)| = (A^3(s) \times Y^3(s) \times \{0\}) \cup (\bar{S}(s) \times Y^3(s) \times R_+ \setminus \{0\}) \cup$$

$$(\bigcup_{\tilde{s} \in P(s)} (A^3(\tilde{s}) \times Y^3(s) \times R_+ \setminus \{0\})) \cup$$

$$(\bigcup_{t \in Q(s)} (A^3(s) \times Y^3(t) \times R_+ \setminus \{0\})) \qquad , s \notin \tau^3(v)$$

and

$$|\delta K^3(s)| = (A^3(s) \times Y^3(s) \times \{0\}) \cup (\bar{S}(s) \times Y^3(s) \times R_+ \setminus \{0\}) \cup$$

$$(A_0^3 \times Y^3(s) \times R_+ \setminus \{0\}) \cup$$

$$(\bigcup_{t \in Q(s)} (A^3(s) \times Y^3(t) \times R_+ \setminus \{0\})) \qquad , s \in \tau^3(v).$$

For each $s \in \tau^3$ we define the function h_s^3: $|K^3(s)| \to U^3(s)$ by

$$h_s^3(x,y,t) = y - tz(x) \qquad , (x,y,t) \in |K^3(s)|$$

where

$$U^3(s) = \{y - tz(x) \in R^{N+M} \mid \forall i \text{ with } I_i^-(s) \subset V_i \text{ and } I_i^+(s) = \emptyset$$
$$\text{holds } x_i^T[y_i - tz_i(x)] = 0\}.$$

Note that the dimension of $U^3(s)$ is $N + M - \sum_{j=1}^{N} k_j(s)$.

<u>Assumption 6.4.</u> The point 0 in R^{N+M} is a regular value of each function h_s^3, $s \in \tau^3$ in the sense of described in the beginning of this section.

Under assumption 5.5 each mapping h_s^3, $s \in \tau^3$, is a piecewise continuously differentiable mapping from $|K^3(s)|$ to $U^3(s)$. Together with assumption 6.4 we obtain from theorem 5.2 that all sets $(h_s^3)^{-1}(0)$,

$s \in \tau^3$, consist of piecewise smooth paths and loops, each path having 0, 1 or 2 end points in $|\delta K^3(s)|$, $s \in \tau^3$.

Let us consider the path $G^3(s^0)$ of $(h^3_{s^0})^{-1}(0)$ starting in the point $(x^0, y^0, t^0) = (v, z(v)/\sum_{(j,k)\in I} z^+_{jk}(v), (\sum_{(j,k)\in I} z^+_{jk}(v))^{-1})$. Observe that this point lies in $|\delta K^3(s^0)|$ since $(x^0, y^0, t^0) \in A^3_0 \times Y^3(s^0) \times R_+\backslash\{0\}$. When $G^3(s^0)$ is unbounded then, because $P^3(s^0)$ is a bounded manifold, at least one of the y_{jk}'s goes to minus infinity or t goes to infinity. In the first case also t goes to infinity since $y = tz(x)$ and because of the boundedness of z on the compact set S. So it follows with $t = (\sum_{(j,k)\in I} z^+_{jk}(x))^{-1}$ that $G^3(s^0)$ approaches in case of unboundedness a limit point (x^*, y^*) in $|L^3(s^0)|$ for which $z(x^*) \leq 0$. If $G^3(s^0)$ is bounded it connects (x^0, y^0, t^0) with another point $(\bar{x}, \bar{y}, \bar{t})$ in $|\delta K^3(s^0)|$, according to theorem 5.2. The point $(\bar{x}, \bar{y}, \bar{t})$ doesn't lie in $A^3_0 \times Y^3(s^0) \times R_+\backslash\{0\}$ because then $(\bar{x}, \bar{y}, \bar{t}) = (x^0, y^0, t^0)$. It is also impossible that $(\bar{x}, \bar{y}, \bar{t})$ lies in $A^3(s^0) \times Y^3(s^0) \times \{0\}$, since then $\bar{y} = 0$ but $0 \notin |D^3(s^0)|$. Suppose $(\bar{x}, \bar{y}, \bar{t})$ to be a point in $\bar{S}(s^0) \times Y^3(s^0) \times R_+\backslash\{0\}$. Because $\bar{y} = \bar{t}z(\bar{x}) \in Y^3(s^0)$ and $\bar{t} > 0$, there exists at least one index $(i,h) \in I$ such that $z_{ih}(\bar{x}) > 0$. From $\bar{x} \in \bar{S}(s^0)$ we know that $\bar{x}_{ip} = 0$ for all $(i,p) \in I^-_i(s^0)$. Therefore $\bar{x}^T_i z_i(\bar{x}) > 0$ yielding a contradiction to the complementarity conditions on z. So, $(\bar{x}, \bar{y}, \bar{t}) \in A^3(s^0) \times Y^3(t) \times R_+\backslash\{0\}$ for some $t \in Q(s^0)$. This cell is also an element of $\delta K^3(t)$ if $t \in \tau^3$. If $t \notin \tau^3$ then $(\bar{x}, \bar{y}, \bar{t})$ lies on the facet $A^3(s^0) \times Y^3(\hat{t}) \times R_+\backslash\{0\}$ of $A^3(s^0) \times Y^3(t) \times R_+\backslash\{0\}$ with the unique $\hat{t} \in \tau^3$ for which $\hat{t} \in Q(t)$ and $s^0 \in P_2(\hat{t})$. This cell is an element of $\delta K^3(\hat{t})$. In this way we connect $G^3(s^0)$ with a path in $|K^3(t)|$ or $|K^3(\hat{t})|$.
For an arbitrary $s \in \tau^3$ the foregoing remains valid. If $s \notin \tau^3(v)$ it is also possible that a path in $(h^3_s)^{-1}(0)$ reaches a facet $A^3(\tilde{s}) \times Y^3(s) \times R_+\backslash\{0\}$, $\tilde{s} \in P(s)$, in $\delta K^3(s)$. From theorem 6.2 we know that $A^3(\tilde{s}) \times Y^3(s) \times R_+\backslash\{0\}$ lies also in $\delta K^3(\tilde{s})$. In this way we link two paths in $|K^3(s)|$ and $|K^3(\tilde{s})|$.

So, starting from $(x^0, y^0, t^0) \in G^3(s^0)$ we get a linked path that jumps from one PDM to another. This path is denoted by G^3. For certain $s \in \tau^3$, $(h^3_s)^{-1}(0)$ is compact if the t-component of the points $(x,y,t) \in$

$(h_s^3)^{-1}(0)$ doesn't go to infinity. So, for each $s \in \tau^3$, the number of bounded paths in $(h_s^3)^{-1}(0)$ is finite. Because the number of sign vectors in τ^3 is finite, the total number of bounded paths in $\bigcup_{s \in \tau^3} (h_s^3)^{-1}(0)$ is finite.

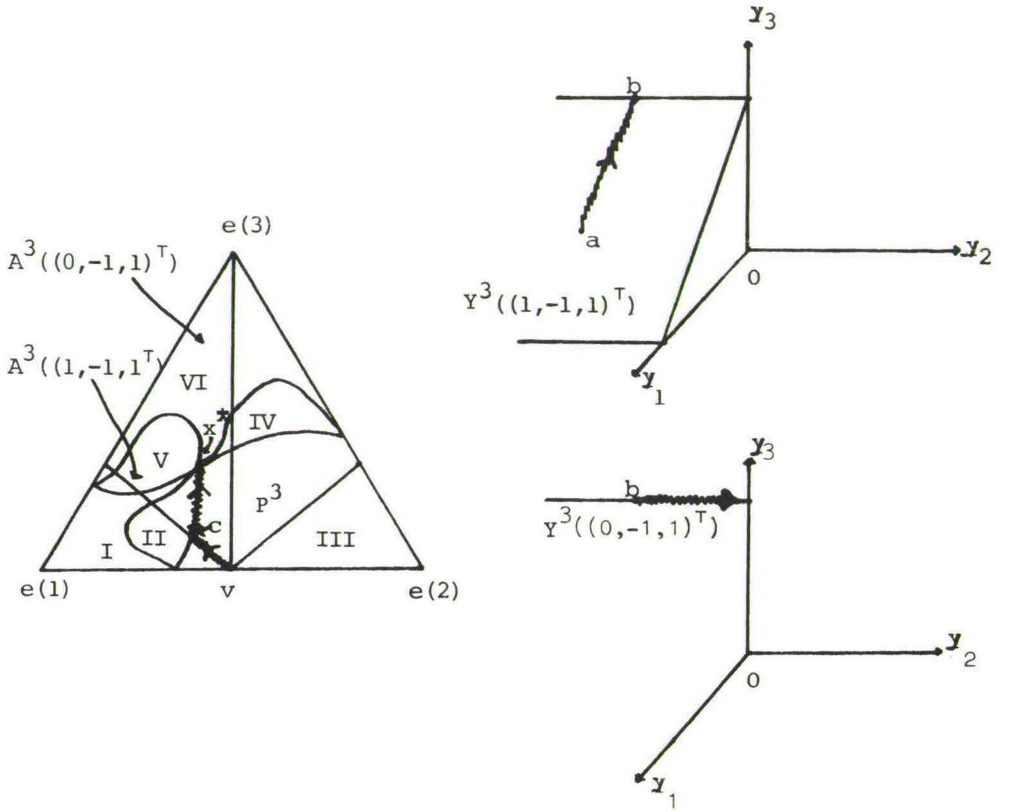
Note that a bounded path in $G^3(s^0)$ can be linked with (x^0,y^0,t^0) and with a path in another PDM whereas each bounded path in $G^3(s)$, $s \neq s^0$, can always be linked with two paths in other PDM's. Therefore G^3 must enter a manifold $|K^3(\bar{s})|$ for which $G^3 \cap |K^3(\bar{s})|$ is unbounded and t goes to infinity. Recall that for all points $(x,y,t) \in G^3$ holds that $t = (\sum_{(j,k)\in I} z_{jk}^+(x))^{-1}$. So, G^3 must approach a limit-point (x^*,y^*) in $|L^3(\bar{s})|$ with $z(x^*) < 0$.

In the foregoing we proved the existence of a path connecting the point $(x^0,y^0) = (v, z(v)/\sum_{(j,k) \in I} z_{jk}^+(v))$ in $|\delta L^3(s^0)|$ with a point (x^*,y^*) in $|L^3(\bar{s})|$, $\bar{s} \in \tau^3$, for which $z(x^*) < 0$. The next corollary states that the path P^3 of the exponent-process corresponds to the projection of G^3 on S, while B^3 is the same projection of $\bigcup_{s \in \tau^3} (h_s^3)^{-1}(0)$.

Corollary 6.5. If $(x,y,t) \in (h_s^3)^{-1}(0)$ for some $s \in \tau^3$, then $x \in B^3(s) = A^3(s) \cap C^3(s)$. More precisely, when $x=v$, $(v,y,t) \in G^3$ with $t = (\sum_{(j,k) \in I} z_{jk}^+(v))^{-1}$ and $y = tz(v) \in Y^3(s^0)$. So, v lies in $B^3(s^0)$. If $x \neq v$ then $(x,y,t) \in A^3(s) \times Y^3(s) \times R_+\backslash\{0\}$, so that $x \in B^3(s)$. Moreover, P^3 is the projection of the path G^3 on S. P^3 exists and connects v with a solution x^* to the NLCP problem on S when the assumptions 5.5 and 6.4 are satisfied.

Remark that the case in which $z(v) < 0$ does not fit in the framework. But then the existence and convergence of P^3 are trivial.

The path G^3 projected on S and the regions $D^3(s)$ are illustrated in figure 6.1 for N=1 and n_1=2.



a. Projection P^3 of G^3 on S

b. Projection of G^3 on $\mathcal{D}^3((+1,-1,+1)^T)$
Projection of G^3 on $\mathcal{D}^3((0,-1,+1)^T)$

Figure 6.1. The projections of G^3 on S, $\mathcal{D}^3((+1,-1,+1)^T)$ and $\mathcal{D}^3((0,-1,+1)^T)$ for N=1 and n_1=2. The projections are heavily drawn. $C^3((-1,+1,+1)^T)$ is denoted by I, $C^3((-1,-1,+1)^T)$ by II, $C^3((+1,-1,+1)^T)$ by III, $C^3((+1,-1,-1)^T)$ by IV, $C^3((-1,+1,-1)^T)$ by V and $C^3((+1,+1,-1)^T)$ by VI. The point a equals $z(v)/(z_1(v)+z_3(v))$ and $b = z(c)/z_3(c)$.

We conclude this section with a short comparison of the different processes by focussing our attention to their respective homotopy-parameter interpretations. For a point (x,y,t) on the path belonging to the sum-process holds $t^{-1} = \max_{(i,j) \in I} z_{ij}(x)$.

When applying the product-process we have $t^{-1} = \sum_{j=1}^{N} \max_{(j,k) \in I(j)} z_{jk}(x)$ and we have $t^{-1} = \sum_{(j,k) \in I} z^{+}_{jk}(x)$ in case of the exponent-process. From this we can conclude that the homotopy-parameter belonging to the exponent-process gives a better measure of the exactness of an approximation. All the positive components of z are considered while the product-process considers only the maximum component of each z_j, $j \in I_N$, and the sum-process only the maximum component of the whole z. All of this together gives theoretical reasons to prefer the exponent-process for searching points $\hat{x}$ with $z(\hat{x}) \leq 0$, where $z : S \to R^{N+M}$ and $x_j^T z_j(x) = 0$ for all $j \in I_N$ and all $x \in S$.

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